

# ON TORUS HOMEOMORPHISMS SEMICONJUGATE TO IRRATIONAL ROTATIONS

T. JÄGER AND A. PASSEGGI

**ABSTRACT.** In the context of the Franks-Misiurewicz Conjecture, we study homeomorphisms of the two-torus semiconjugate to an irrational rotation of the circle. As a special case, this conjecture asserts uniqueness of the rotation vector in this class of systems. We first characterise these maps by the existence of an invariant ‘foliation’ by essential annular continua (essential subcontinua of the torus whose complement is an open annulus) which are permuted with irrational combinatorics. This result places the considered class close to skew products over irrational rotations. Generalising a well-known result of M. Herman on forced circle homeomorphisms, we provide a criterion, in terms of topological properties of the annular continua, for the uniqueness of the rotation vector.

As a byproduct, we obtain a simple proof for the uniqueness of the rotation vector on decomposable invariant annular continua with empty interior. In addition, we collect a number of observations on the topology and rotation intervals of invariant annular continua with empty interior.

*2010 Mathematics Subject Classification.* Primary 54H20, Secondary 37E30, 37E45

## 1. INTRODUCTION

Rotation Theory, as a branch of dynamical systems, goes back to Poincaré’s celebrated classification theorem for circle homeomorphisms. It states that given an orientation-preserving circle homeomorphism  $f$  with lift  $F : \mathbb{R} \rightarrow \mathbb{R}$ , the limit

$$\rho(F) = \lim_{n \rightarrow \infty} (F^n(x) - x)/n,$$

called the *rotation number* of  $f$ , exists and is independent of  $x$ . Furthermore,  $\rho(F)$  is rational if and only if there exists a periodic orbit and  $\rho(F)$  is irrational if and only if  $f$  is semiconjugate to an irrational rotation.

Since both cases of the above dichotomy are easy to analyse, this result provides a complete description of the possible long-term behaviour for a whole class of systems without any additional *a priori* assumptions – a situation which is still rare even nowadays in the theory of dynamical systems. In addition, the rotation number can be viewed as an element of the first homological group of the circle and thus provides a link between the dynamical behaviour of homeomorphisms and the topological structure of the manifold. It is not surprising that the consequences of this result have found numerous applications in the sciences, ranging from quantum physics to neural biology [1, 2]. Hence, the attempt to apply this approach to higher-dimensional manifolds, in order to obtain a classification of possible dynamics in terms of rotation vectors and rotation sets, is most natural. However, despite impressive contributions over the last decades, fundamental problems still remain open even in dimension two.

Already in the case of the two-dimensional torus  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ , a unique rotation vector does not have to exist. Instead, given a torus homeomorphism  $f$  homotopic to the identity and a lift  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , the *rotation set* is defined as

$$\rho(F) = \left\{ \rho \in \mathbb{R}^2 \mid \exists z_i \in \mathbb{R}^2, n_i \nearrow \infty : \lim_{i \rightarrow \infty} (F^{n_i}(z_i) - z_i)/n_i = \rho \right\}.$$

This is always a compact and convex subset of the plane [3]. Consequently, three principal cases can be distinguished according to whether the rotation set (1) has non-empty interior,

---

Department of mathematics, TU-Dresden. Email addresses: Tobias.Oertel-Jaeger@tu-dresden.de, alepasseggi@gmail.com.

(2) is a line segment of positive length or (3) is a singleton, that is,  $f$  has a unique rotation vector. Existing results on each of the three cases suggest that a classification approach is indeed feasible: for example, in case (1) the dynamics are ‘rich and chaotic’, in the sense that the topological entropy is positive [4] and all the rational rotation vectors in the interior of  $\rho(F)$  are realised by periodic orbits [5]; in case (3) a Poincaré-like classification exists under the additional assumption of area-preservation and a certain *bounded mean motion* property [6], and the consequences of *unbounded mean motion* are being explored recently as well [7, 8, 9]. In case the rotation set is a segment of positive length, examples can be constructed whose rotation set is either (a) a segment with rational slope and infinitely many rational points or (b) a segment with irrational slope and one rational endpoint [10]. Recent results on torus homeomorphisms with this type of rotation segments indicate that these examples can be seen as good models for the general case [11, 12]. In addition, there exist many further results that provide more information on each of the three cases. Just to mention some of the important contributions in this direction, we refer to [13, 14, 15, 16, 17, 18].

In the light of these advances, it seems reasonable to say that the outline of a complete classification emerges. Yet, there is still a major blank spot in the current state of knowledge. It is not known whether any rotation segment other than the two cases (a) and (b) mentioned above can occur. Actually, it was conjectured by Franks and Misiurewicz in [10] that this cannot happen. However, while this conjecture has been in the focus of attention for more than two decades, it has defied all experts and up to date there are still only very partial results on the problem. A deeper reason for this may lie in the fact that it concerns dynamics without any periodic points – the main task is to exclude the existence of rotation segments without rational points<sup>1</sup> – and therefore many standard techniques in topological dynamics based on the existence of periodic orbits fail to apply. Hence, it is likely that proving (or disproving) the conjecture will require to obtain a much better understanding of periodic-point free dynamics, which seems a worthy task in a much broader context as well.

We believe that in this situation the systematic investigation of suitable subclasses of periodic point free torus homeomorphisms is a good way to obtain further insight. In fact, there are some classes that have been studied intensively already. First, Franks and Misiurewicz proved that the conjecture is true for time-one maps of flows [10]. Secondly, Kwapisz considered torus homeomorphisms that preserve the leaves of an irrational foliation and showed that the rotation set is either a segment with a rational endpoint or a singleton [19]. Finally, for skew products over irrational rotations on the torus, Herman proved the uniqueness of the rotation vector [20]. Hence, in both cases the conjecture was confirmed for the particular subclasses, which are certainly very restrictive compared to general torus homeomorphisms. However, since these are the only existing partial results on the problem, they are the only obvious starting point for further investigations. The aim of this article is to make a first step in this direction by studying torus homeomorphisms which are semiconjugate to a one-dimensional irrational rotation. For obvious reasons these do not have any periodic orbits, but apart from this little is known about the dynamical implications of this property. We first provide an analogous characterisation of these systems.

Denote by  $\text{Homeo}_0(\mathbb{T}^d)$  the set of homeomorphisms of the  $d$ -dimensional torus that are homotopic to the identity. Recall that an *essential annular continuum*  $A \subseteq \mathbb{T}^2$  is a continuum whose complement  $\mathbb{T}^2 \setminus A$  is homeomorphic to the open annulus  $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$ . An *essential circloid* is an essential annular continuum which is minimal with respect to inclusion amongst all essential annular continua. We refer to Section 2 for the corresponding definitions in higher dimensions. Note that for any family of pairwise disjoint essential continua in  $\mathbb{T}^d$  there exists a natural circular order. We say a wandering<sup>2</sup> essential continuum has irrational combinatorics (with respect to  $f \in \text{Homeo}_0(\mathbb{T}^d)$ ) if its orbit is ordered in  $\mathbb{T}^d$  in the same way as the orbit of an irrational rotation on  $\mathbb{T}^1$ . See Section 3 for more details.

**Theorem 1.** *Suppose  $f \in \text{Homeo}_0(\mathbb{T}^d)$ . Then the following statements are equivalent.*

<sup>1</sup>Note that a periodic orbit always has a rational rotation vector.

<sup>2</sup>We call  $A \subseteq \mathbb{T}^d$  *wandering*, if  $f^n(A) \cap A = \emptyset \ \forall n \geq 1$ .

- (i)  $f$  is semiconjugate to an irrational rotation  $R$  of the circle;
- (ii) there exists a wandering essential circloid with irrational combinatorics;
- (iii) there exists a wandering essential continuum with irrational combinatorics;
- (iv) there exists a semiconjugacy  $h$  from  $f$  to  $R$  such that for all  $\xi \in \mathbb{T}^1$  the fibre  $h^{-1}\{\xi\}$  is an essential annular continuum.

The proof is given in Section 3. Issues concerning the uniqueness of the semiconjugacy in the above situation are discussed in Section 4. In general the semiconjugacy is not unique, but there exist important situations where it is unique up to post-composition with a rotation. In this case every semiconjugacy has only essential annular continua as fibres.

For the two-dimensional case, the implication “(iii)  $\Rightarrow$  (i)” in Theorem 1 is contained in [21], and the proof easily extends to higher dimensions. In our context, the most important fact will be the equivalence “(i)  $\Leftrightarrow$  (iv)”, which says that the semiconjugacy can always be chosen such that its fibres are annular continua. This places the considered systems very close to skew products over irrational rotations, with the only difference that the topological structure of the fibres can be more complicated. For this reason, one may hope to generalise Herman’s result to this larger class of systems, thus proving the existence of a unique rotation vector. To that end, however, we here have to make an additional assumption on topological regularity the fibres of the semiconjugacy.

An essential annular continuum  $A \subseteq \mathbb{T}^2$  admits essential simple closed curves in its complement. The homotopy type of such curves is unique, and we define it to be the *homotopy type of  $A$* . We say  $A$  is *horizontal* if its homotopy type is  $(1, 0)$ . Given a horizontal annular continuum  $A$ , we denote by  $\hat{A}$  a connected component of  $\pi^{-1}(A)$ , where  $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2$  is the canonical projection. Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $(x, y) \mapsto (x+1, y)$ . Then we say  $A$  is *compactly generated* if there exists a compact connected set  $G_0 \subseteq \hat{A}$  such that  $A = \bigcup_{n \in \mathbb{Z}} T^n(G_0)$ . An essential annular continuum with arbitrary homotopy type is said to be compactly generated if there exists a homeomorphism of  $\mathbb{T}^2$  which maps it to a compactly generated horizontal one.

**Theorem 2.** *Suppose  $f \in \text{Homeo}_0(\mathbb{T}^2)$  is semiconjugate to an irrational rotation of the circle. Further, assume that the semiconjugacy  $h$  is chosen such that its fibres  $h^{-1}(\xi)$  are all essential annular continua and there exists a set  $\Omega \subseteq \mathbb{T}^1$  of positive Lebesgue measure such that  $h^{-1}\{\xi\}$  is compactly generated for all  $\xi \in \Omega$ . Then  $f$  has a unique rotation vector.*

The proof is given in Section 5. We say an annular continuum is *thin*, if it has empty interior. Note that in the situation of Theorem 2, all but at most countably many of the fibres are thin in this sense.

As a byproduct of our methods, we also obtain the uniqueness of the rotation vector for invariant compactly generated thin annular continua.

**Theorem 3.** *Suppose  $f \in \text{Homeo}_0(\mathbb{T}^2)$  and  $A$  is a thin annular continuum that is compactly generated and  $f$ -invariant. Then  $f|_A$  has a unique rotation vector, that is, there exists a vector  $\rho \in \mathbb{R}^2$  such that  $\lim_{n \rightarrow \infty} (F^n(z) - z/n) = \rho$  for all  $z \in \mathbb{R}^2$  with  $\pi(z) \in A$ . Moreover, the convergence is uniform in  $z$ .*

For decomposable<sup>3</sup> circloids the above statement was proved before by Le Calvez [22], using Caratheodory’s prime ends, which is a classical approach to study the rotation theory of continua [23, 24, 25]. It should be noted that there exist important examples of invariant thin annular continua which are not compactly generated. One example is the Birkhoff attractor [26], which does not have a unique rotation vector and therefore cannot have compact generator due to the above statement. Another well-known example is the pseudo-circle, which was constructed by Bing in [27] and latter shown to occur as a minimal set of smooth surface diffeomorphisms [28, 29]. Whether pseudo-circles admit dynamics with non-unique rotation vectors is still open.

<sup>3</sup>A continuum is called *decomposable*, if it can be written as the union of two strict subcontinua.

We close by collecting some observations on the topology and dynamics of invariant thin annular continua in Section 6. It is known that any thin annular continuum  $A$  contains a unique circloid  $C_A$  (see Lemma 2.2). We show that if  $A$  is compactly generated, then so is the circloid  $C_A$ . Conversely, if  $C_A$  is compactly generated then either  $A$  is compactly generated as well or  $A$  contains at least one *infinite spike*, that is, an unbounded connected component of  $A \setminus C_A$ . Finally, reproducing some examples due to Walker [30] we show that thin annular continua can have any compact interval as rotation segment, even in the absence of periodic orbits.

**Acknowledgements.** This work was supported by an Emmy-Noether grant of the German Research Council (DFG grant JA 1721/2-1). We would like to thank Patrice Le Calvez for helpful comments and Henk Bruin for fruitful discussions leading to the remarks on the uniqueness of the semiconjugacies in Section 4.

## 2. NOTATION AND PRELIMINARIES

The following notions are usually used in the study of dynamics on the two-dimensional torus or annulus. For convenience, we stick to the same terminology also in higher dimensions. We let  $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$  and denote by  $\mathbb{A}^d = \mathbb{T}^{d-1} \times \mathbb{R}$  the  $d$ -dimensional annulus. If  $d = 2$ , we simply write  $\mathbb{A}$  instead of  $\mathbb{A}^2$ . We will often compactify  $\mathbb{A}$  by adding two points  $-\infty$  and  $+\infty$ , thus making it a sphere. As long as no ambiguities can arise, we will always denote canonical quotient maps like  $\mathbb{R} \rightarrow \mathbb{T}^1$ ,  $\mathbb{R}^d \rightarrow \mathbb{T}^d$ ,  $\mathbb{R}^d \rightarrow \mathbb{A}^d$  by  $\pi$ . Likewise, on any product space  $\pi_i$  denotes the projection to the  $i$ -th coordinate. We call a subset  $A \subseteq \mathbb{A}^d$  or  $A \subseteq \mathbb{R}^d$  bounded from above (from below) if  $\pi_d(A)$  is bounded from above (from below).

We say a continuum (that is, a compact and connected set)  $E \subseteq \mathbb{A}^d$  is *essential* if  $\mathbb{A}^d \setminus E$  contains two unbounded connected components. In this case, one of these components will be unbounded above and bounded below, and we denote it by  $\mathcal{U}^+(A)$ . The second unbounded component will be bounded above and unbounded below, and we denote it by  $\mathcal{U}^-(A)$ .  $A$  is called an *essential annular continuum* if  $\mathbb{A}^d \setminus A = \mathcal{U}^+(A) \cup \mathcal{U}^-(A)$ . Note that in dimension two, one can show by using the Riemann Mapping Theorem that both unbounded components are homeomorphic to  $\mathbb{A}$  and  $A$  is the intersection of a decreasing sequence of topological annuli. This is not true anymore in higher dimensions, but at least we have the following.

**Lemma 2.1.** *If  $(A_n)_{n \in \mathbb{N}}$  is a decreasing sequence of essential annular continua, then  $A = \bigcap_{n \in \mathbb{N}} A_n$  is an essential annular continuum as well.*

*Proof.* As a decreasing intersection of essential continua,  $A$  is an essential continuum. Further, we have that  $\mathbb{T}^d \setminus A = \bigcup_{n \in \mathbb{N}} \mathbb{T}^d \setminus A_n$  is the union of the two sets

$$\mathcal{U}^+ = \bigcup_{n \in \mathbb{N}} \mathcal{U}^+(A_n) \quad \text{and} \quad \mathcal{U}^- = \bigcup_{n \in \mathbb{N}} \mathcal{U}^-(A_n).$$

As the union of an increasing sequence of open connected sets is connected, both these sets are connected. Hence,  $\mathbb{T}^d \setminus A$  consists of exactly two connected components  $\mathcal{U}^+(A) = \mathcal{U}^+$  and  $\mathcal{U}^-(A) = \mathcal{U}^-$ , both of which are unbounded.  $\square$

Given  $S \subseteq \mathbb{R}^d$ , we say  $S$  is *horizontal* if  $\pi_d(S)$  is bounded and  $\mathbb{R}^d \setminus S$  contains two different connected components  $\mathcal{U}^+(S)$  and  $\mathcal{U}^-(S)$  whose image under  $\pi_d$  is unbounded. Note that in this case one of the two components, which we always denote by  $\mathcal{U}^+(S)$ , is bounded below whereas the other component, denoted by  $\mathcal{U}^-(S)$ , is bounded above. Similarly, given any subset  $B \subseteq \mathbb{A}^d$  bounded above (below) we denote by  $\mathcal{U}^+(B)$  ( $\mathcal{U}^-(B)$ ) the unique connected component of  $\mathbb{A}^d \setminus B$  which is unbounded above (below). The same notation is used on  $\mathbb{R}^d$ . A horizontal connected closed set  $S$  is called a *horizontal strip*, if  $\mathbb{R}^d \setminus S = \mathcal{U}^+(S) \cup \mathcal{U}^-(S)$ . Note that thus the lift of an essential annular continuum  $A \subseteq \mathbb{A}^d$  to  $\mathbb{R}^d$  is a horizontal strip.

In any  $d$ -dimensional manifold  $M$ , we say  $A$  is an *annular continuum* if it is contained in a topological annulus  $\mathcal{A} \simeq \mathbb{A}^d$  and it is an essential annular continuum in the above sense when viewed as a subset of  $\mathcal{A}$ . In this situation, we say  $A$  is *essential* if essential loops in

$\mathcal{A}$  are also essential in  $M$ . We call  $C \subseteq \mathbb{A}^d$  an *essential circloid* if it is an essential annular continuum and does not contain any other essential annular continuum as a strict subset. Circloids in general manifolds are then defined in the same way as annular continua. Finally, we call a horizontal strip  $S$  *minimal* if it is a minimal element of the set of horizontal strips with the partial ordering by inclusion. Note that thus an annular essential continuum in  $\mathbb{A}^d$  is a circloid if and only if its lift to  $\mathbb{R}^d$  is a minimal strip. Finally, we call a closed set *thin*, if it has empty interior.

**Lemma 2.2** ([6], Lemma 3.4). *Every thin annular continuum  $A$  contains a unique circloid  $C_A$ , which is given by*

$$(2.1) \quad C_A = \overline{\mathcal{U}^+(A)} \cap \overline{\mathcal{U}^-(A)}.$$

*The same statement applies to thin horizontal strips.*

The proof in [6] is given for essential annular continua and for  $d = 2$ , but it literally goes through in higher dimensions and for strips. The same is true for the following result, which describes an explicit construction to obtain essential circloids from arbitrary essential continua. Given an essential set  $A \subseteq \mathbb{R}^d$  which is bounded above, we write  $\mathcal{U}^{+-}(A)$  instead of  $\mathcal{U}^-(\mathcal{U}^+(A))$  and use analogous notation for other concatenations of these procedures.

**Lemma 2.3** ([6], Lemma 3.2). *If  $A \subseteq \mathbb{A}^d$  is an essential continuum, then*

$$\mathcal{C}^+(A) = \mathbb{T}^d \setminus (\mathcal{U}^{+-}(A) \cup \mathcal{U}^{++}(A)) \quad \text{and} \quad \mathcal{C}^-(A) = \mathbb{T}^d \setminus (\mathcal{U}^{-+}(A) \cup \mathcal{U}^{--}(A))$$

*are circloids. Further, we have  $\partial \mathcal{C}^\pm(A) \subseteq A$ .*

Given two horizontal essential continua  $A_1, A_2 \subseteq \mathbb{T}^d$ , we say  $\hat{A}_i \subseteq \mathbb{A}^d$  is a lift of  $A_i$  if it is a connected component of  $\pi^{-1}(A_i)$ . We write  $\hat{A}_1 \preccurlyeq \hat{A}_2$  if  $\hat{A}_2 \subseteq \mathcal{U}^+(\hat{A}_1)$ . Given two lifts  $\hat{A}_1 \preccurlyeq \hat{A}_2$  we say they are *adjacent*, if if the compact region  $\mathbb{A}^d \setminus (\mathcal{U}^-(\hat{A}_1) \cup \mathcal{U}^+(\hat{A}_2))$  does not contain any integer translates of  $\hat{A}_1$  and  $\hat{A}_2$ . In this situation, we let  $[\hat{A}_1, \hat{A}_2] = \mathbb{A}^d \setminus (\mathcal{U}^-(\hat{A}_1) \cup \mathcal{U}^+(\hat{A}_2))$  and  $(\hat{A}_1, \hat{A}_2) = \mathcal{U}^+(\hat{A}_1) \cap \mathcal{U}^-(\hat{A}_2)$ . Then we let  $[A_1, A_2] = \pi([\hat{A}_1, \hat{A}_2])$  and  $(A_1, A_2) = \pi((\hat{A}_1, \hat{A}_2))$ . With these notions, we define a circular order on pairwise disjoint essential continua  $A_1, A_2, A_3 \subseteq \mathbb{T}^d$  by

$$A_1 \preccurlyeq A_2 \preccurlyeq A_3 \quad \Leftrightarrow \quad A_2 \in [A_1, A_3].$$

The strict relation  $\prec$  is defined by replacing closed *intervals* with open ones. Using these notions, we now say a sequence  $(A_n)_{n \in \mathbb{N}}$  of pairwise disjoint essential continua in  $\mathbb{T}^d$  has *irrational combinatorics* if there exists  $\rho \in \mathbb{R} \setminus \mathbb{Q}$  such that for arbitrary  $y_0 \in \mathbb{T}^1$  the sequence  $y_n = y_0 + n\rho \bmod 1$  satisfies

$$A_k \prec A_m \prec A_n \quad \Leftrightarrow \quad y_k < y_m < y_n$$

for all  $k, m, n \in \mathbb{Z}$ .

We finish the topological preliminaries with the following proposition.

**Proposition 2.4.** *Let  $A, B$  be compact subsets of  $\mathbb{A}^n$  such that:*

- $A \cap B = \emptyset$
- $\mathbb{A}^n \setminus A$  has exactly one unbounded component.
- $\mathbb{A}^n \setminus B$  has exactly one unbounded component.

*Then  $\mathbb{A}^n \setminus A \cup B$  has one exactly unbounded component.*

*Proof.* Let  $\Pi_{0,b}(A^c), \Pi_{0,b}(B^c)$  be the sets of bounded connected components of  $A^c, B^c$  respectively. We define

$$\text{Fill}(A) = A \cup \left[ \bigcup_{U \in \Pi_{0,b}(A^c)} U \right], \quad \text{Fill}(B) = B \cup \left[ \bigcup_{U \in \Pi_{0,b}(B^c)} U \right]$$

Hence  $A \subset \text{Fill}(A)$ ,  $B \subset \text{Fill}(B)$  and  $\text{Fill}(A)^c, \text{Fill}(B)^c$  have each only one connected component which is unbounded. Furthermore, we may choose sets  $C, D \subseteq \mathbb{A}^n$  such that  $\text{Fill}(A) \cup \text{Fill}(B) = C \cup D$  and we have that

- $C, D$  are compact;
- $C \cap D = \emptyset$ ;
- $C^c, D^c$  has each only one connected component.

Notice that one of the two sets  $C$  or  $D$  could be empty.

Now, it suffices to show that  $(C \cup D)^c$  has only one connected component. For this we consider the usual compactification of the annulus by the sphere  $\mathbb{S}^n$ , and the unreduced Mayer-Vietoris long sequence (see [31]) for  $(\mathbb{S}^n, C^c, D^c)$  starting from  $H_1(\mathbb{S}^n)$ :

$$H_1(\mathbb{S}^n) \xrightarrow{\partial_*} H_0(C^c \cap D^c) \xrightarrow{\theta_*} H_0(C^c) \oplus H_0(D^c) \xrightarrow{\xi_*} H_0(\mathbb{S}^n) \rightarrow 0$$

Then, we obtain that

$$0 \xrightarrow{\partial_*} H_0(C^c \cap D^c) \xrightarrow{\theta_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\xi_*} \mathbb{Z} \rightarrow 0$$

Thus we have that  $H_0(C^c \cap D^c)$  is isomorphic to  $\ker(\xi_*)$ , so  $H_0(C^c \cap D^c) \cong \mathbb{Z}$ . This implies that there is only one connected component in  $C^c \cap D^c$ .  $\square$

Finally, we will frequently use the following Uniform Ergodic Theorem (e.g. [32, 33]).

**Theorem 2.5.** *Suppose  $X$  is a compact metric space and  $f : X \rightarrow X$  and  $\varphi : X \rightarrow \mathbb{R}$  are continuous. Further, assume that there exists  $\rho \in \mathbb{R}$  such that*

$$\int_X \varphi \, d\mu = \rho$$

*for all  $f$ -invariant ergodic probability measures  $\mu$  on  $X$ . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{i=0}^{n-1} \varphi \circ f^i(x) \right) = \rho \quad \text{for all } x \in X.$$

*Furthermore, the convergence is uniform on  $X$ .*

### 3. SEMICONJUGACY TO AN IRRATIONAL ROTATION

We now turn to the proof of Theorem 1. The implications (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i) in Theorem 1 are obvious. Hence, in order to prove all the equivalences, it suffices to prove (iii) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv). We do so in three separate lemmas and start by treating the easiest of the three implications, which is (iii) $\Rightarrow$ (ii).

**Lemma 3.1.** *Let  $f \in \text{Homeo}_0(\mathbb{T}^d)$  and suppose  $E$  is a wandering essential continuum. Then  $C^+(E)$  is a wandering essential circloid and the circular ordering of the orbits of  $E$  and  $C^+(E)$  are the same.*

*Proof.* Suppose  $f \in \text{Homeo}_0(\mathbb{T}^d)$  and  $E$  is a wandering essential continuum with irrational combinatorics. Let  $E_n = f^n(E)$  and  $C_n = C^+(E_n) = f^n(C^+(E))$ . Assume for a contradiction that the  $C_n$  are not pairwise disjoint, that is,  $C_i \cap C_j \neq \emptyset$  for some integers  $i \neq j$ . Since  $\partial C_n \subseteq E_n$  for all  $n \in \mathbb{Z}$ ,  $C_i$  must intersect the interior of  $C_j$  or vice versa. Assuming the first case, by connectedness  $C_i$  has to be contained in a single connected component of  $\mathbb{T}^d \setminus \partial C_j$ , since otherwise it would have to intersect  $\partial C_j$ . As  $C_i$  is not contained in  $\mathbb{T}^d \setminus C_j$ , it has to be contained in the interior of  $C_j$ . However, this contradicts the minimality of circloids with respect to inclusion. Hence  $C_0$  is wandering.

The circular ordering is preserved when going from  $(E_n)_{n \in \mathbb{N}}$  to  $(C_n)_{n \in \mathbb{N}}$  is now obvious.  $\square$

The next lemma shows (ii) $\Rightarrow$ (iv).

**Lemma 3.2.** *Let  $f \in \text{Homeo}_0(\mathbb{T}^d)$  and suppose  $C$  is a wandering essential circloid with irrational combinatorics of type  $\rho$ . Then there exists a semiconjugacy  $h : \mathbb{T}^d \rightarrow \mathbb{T}^1$  to  $R_\rho$  such that the fibres  $h^{-1}\{\xi\}$  are all essential annular continua.*

*Proof.* We let  $C_n := f^n(C)$  and denote the connected components of the lifts of these circloid by  $\hat{C}_{n,m}$ , where the indices are chosen such that for all integers  $n, m$  we have

- $\pi(\hat{C}_{n,m}) = C_n$ ;

- $F(\hat{C}_{n,m}) = \hat{C}_{n+1,m};$
- $T(\hat{C}_{n,m}) = \hat{C}_{n,m+1}.$

We claim that

$$H(z) = \sup \left\{ n\rho + m \mid z \in \mathcal{U}^+(\hat{C}_{n,m}) \right\}$$

is a lift of a semiconjugacy  $h$  with the required properties. Note that due to the irrational combinatorics we have  $n\rho + m \leq \tilde{n}\rho + \tilde{m}$  if and only if  $\hat{C}_{n,m} \preceq \hat{C}_{\tilde{n},\tilde{m}}$ , such that in particular  $H(z)$  is well-defined and finite for all  $z \in \mathbb{A}^d$ . Further, for any  $z \in \mathbb{A}^d$  we have

$$\begin{aligned} H \circ F(z) &= \sup \left\{ n\rho + m \mid F(z) \in \mathcal{U}^+(\hat{C}_{n,m}) \right\} \\ &= \sup \left\{ n\rho + m \mid z \in \mathcal{U}^+(\hat{C}_{n-1,m}) \right\} = H(z) + \rho. \end{aligned}$$

In a similar way one can see that  $H \circ T(z) = H(z) + 1$ , such that  $H$  projects to a map  $h : \mathbb{T}^d \rightarrow \mathbb{T}^1$  which satisfies  $h \circ f = R_\rho \circ h$ .

In order to check the continuity of  $H$ , suppose  $U \subseteq \mathbb{R}$  is an open interval and let  $z \in H^{-1}(U)$ . Choose  $r = n\rho + m < H(z) < \tilde{n}\rho + \tilde{m} = s$  with  $r, s \in U$ . Then  $z \in \mathcal{U}^+(\hat{C}_{n,m}) \cap \mathcal{U}^-(\hat{C}_{\tilde{n},\tilde{m}}) =: V$ . From the definition of  $H$  we see that  $H(V) \subseteq [r, s] \subseteq U$ , and thus  $H^{-1}(U)$  contains an open neighbourhood of  $z$ . Since  $U$  and  $z \in H^{-1}(U)$  were arbitrary,  $H$  is continuous. The fact that  $h$  is onto follows easily from the minimality of  $R_\rho$ , so that  $h$  is indeed a semiconjugacy from  $f$  to  $R_\rho$ .

It remains to prove the fact that the fibres  $h^{-1}\{\xi\}$  are annular continua. In order to do so, note that for  $\xi \in \mathbb{T}^1$

$$\begin{aligned} (3.1) \quad H^{-1}\{\xi\} &= \bigcap_{n\rho+m<\xi} \mathcal{U}^+(\hat{C}_{n,m}) \cap \bigcap_{\tilde{n}\rho+\tilde{m}>\xi} \mathcal{U}^-(\hat{C}_{\tilde{n},\tilde{m}}) \\ &= \bigcap_{n\rho+m<\xi} \mathbb{A}^d \setminus \mathcal{U}^-(\hat{C}_{n,m}) \cap \bigcap_{\tilde{n}\rho+\tilde{m}>\xi} \mathbb{A}^d \setminus \mathcal{U}^+(\hat{C}_{\tilde{n},\tilde{m}}). \end{aligned}$$

Note here that for all  $n, m, n', m'$  with  $n\rho + m < n'\rho + m'$  we have

$$\mathcal{U}^+(\hat{C}_{n',m'}) \subseteq \mathbb{A}^d \setminus \mathcal{U}^-(\hat{C}_{n',m'}) \subseteq \mathcal{U}^+(\hat{C}_{n,m})$$

and similar inclusions hold in the opposite direction. This explains

the second equality in (3.1). Choosing sequences  $n_i, m_i, \tilde{n}_i, \tilde{m}_i$  with  $n_i\rho + m_i \nearrow \xi$  and  $\tilde{n}_i\rho + \tilde{m}_i \searrow \xi$ , we can rewrite (3.1) as

$$H^{-1}\{\xi\} = \bigcap_{i \in \mathbb{N}} \mathbb{A}^d \setminus \left( \mathcal{U}^-(\hat{C}_{n_i, m_i}) \cup \mathcal{U}^+(\hat{C}_{\tilde{n}_i, \tilde{m}_i}) \right).$$

Since the sets of the intersection are all essential annular continua, so is  $H^{-1}\{\xi\}$  by Lemma 2.1.  $\square$

It remains to prove the implication (i) $\Rightarrow$ (iii).

**Lemma 3.3.** *Suppose  $h : \mathbb{T}^d \rightarrow \mathbb{T}^1$  is a semiconjugacy from  $f \in \text{Homeo}_0(\mathbb{T}^d)$  to an irrational rotation  $R_\rho$ . Then every fibre  $h^{-1}\{\xi\}$  contains a wandering essential continuum with irrational combinatorics.*

*Proof.* We first show that the action  $h^* : \Pi_1(\mathbb{T}^d) \rightarrow \Pi_1(\mathbb{T}^1)$  of  $h$  on the fundamental groups is non-trivial. Suppose for a contradiction that  $h^* = 0$ . Then any lift  $H : \mathbb{R}^d \rightarrow \mathbb{R}$  of  $h$  is bounded since in this case  $\sup_{z \in \mathbb{R}^d} \|H(z)\| = \sup_{z \in [0,1]^d} \|H(z)\|$ . However, this contradicts the unboundedness of

$$H \circ F^n(z) = R_\rho^n \circ H(z).$$

Consequently,  $h^*$  is non-trivial, and by composing  $h$  with a linear torus automorphism we may assume that  $h^*$  is just the projection to the last coordinate. This composition may change the rotation number, but does not effect its irrationality. We obtain a lift  $\hat{h} : \mathbb{A}^d \rightarrow \mathbb{R}$  which satisfies  $\hat{h}(z) \rightarrow \pm\infty$  if  $z \rightarrow \pm\infty$ .

As a consequence, the Intermediate Value Theorem implies that every properly embedded line  $\Gamma = \{\gamma(t) \mid t \in \mathbb{R}\}$  intersects all level sets  $\hat{E}_x = \hat{h}^{-1}\{x\}$ . Hence, all  $\hat{E}_x$  are essential.

If  $\widehat{E}_x$  is not connected, we consider the family of all compact and essential subsets of  $\widehat{E}_x$  and choose an element  $\widehat{\mathcal{E}}$  which is minimal with respect to the inclusion. Note that such minimal elements exist by the Lemma of Zorn. By Proposition 2.4  $\widehat{\mathcal{E}}$  is connected. Further,  $\mathcal{E} = \pi(\widehat{\mathcal{E}})$  is wandering since  $\widehat{\mathcal{E}} \subseteq h^{-1}\{x\}$ . Hence,  $\mathcal{E}$  is the wandering essential continuum we are looking for. The fact that  $\mathcal{E}$  has irrational combinatorics can be seen from the semi-conjugacy equation.  $\square$

#### 4. ON THE UNIQUENESS OF THE SEMICONJUGACY.

In the light of the preceding section, it is an obvious question to ask to what extent a semiconjugacy between  $f \in \text{Homeo}(\mathbb{T}^2)$  and an irrational rotation  $R_\rho$  of the circle is unique. It is easy to check that for every rigid rotation  $R : \mathbb{T}^1 \rightarrow \mathbb{T}^1$  the map  $R \circ h$  is a semiconjugacy between  $f$  and  $R_\rho$  as well. Hence there is non-uniqueness of the semiconjugacy in general. Nevertheless, one could ask whether there is uniqueness up to post-composition with rotations. In brief, we will speak of *uniqueness modulo rotations*.

Consider  $f \in \text{Homeo}_0(\mathbb{T}^2)$  given by  $f(x, y) = (x + \rho_1, y)$  with  $\rho_1 \in \mathbb{Q}^c$ . For any continuous function  $\alpha : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ , we have that  $h_\alpha(x, y) = x + \alpha(y)$  is a semiconjugacy from  $f$  to  $R_{\rho_1}$ . Thus we do not have uniqueness of the semiconjugacy even modulo rotations. However, it is not difficult to see that all the possible semiconjugacies between  $f$  and  $R_{\rho_1}$  are given by  $h_\alpha$  for some continuous function  $\alpha$ . This implies in particular that on every minimal set  $Y_r = \{(x, y) \in \mathbb{T}^2 \mid y = r\}$ ,  $r \in \mathbb{T}^1$ , given any two semiconjugacies  $h_1$  and  $h_2$  we have that  $h_1|_{Y_r} = (R \circ h_2)|_{Y_r}$  for some rigid rotation  $R$ . This, as we will see, is a general fact.

We say that an  $f$ -invariant set  $\Omega$  is *externally transitive* if for every  $x, y \in \Omega$  and neighbourhoods  $U_x, U_y$  of  $x$  and  $y$ , respectively, there exists  $n \in \mathbb{N}$  such that  $f^n(U_x) \cap U_y \neq \emptyset$ . Notice that  $f^n(U_x)$  and  $U_y$  do not need to intersect in  $\Omega$  as in the usual definition of topological transitivity. In the above example the sets  $Y_r$  are transitive, hence externally transitive.

Given  $f \in \text{Homeo}_0(\mathbb{T}^2)$  semiconjugate to a rigid rotation  $R_\rho$  and a  $f$ -invariant set  $\Omega \subseteq \mathbb{T}^2$ , we say the semiconjugacy is *unique modulo rotations on  $\Omega$*  if for all semiconjugacies  $h_1, h_2$  from  $f$  to  $R_\rho$  we have  $h_1|_\Omega = (R \circ h_2)|_\Omega$  for some rigid rotation  $R$ .

**Proposition 4.1.** *Let  $f \in \text{Homeo}(\mathbb{T}^2)$  be semiconjugate to a rigid rotation of  $\mathbb{T}^1$ . Further, assume that  $\Omega \subset \mathbb{T}^2$  is an externally transitive set of  $f$ . Then the semiconjugacy is unique modulo rotations on  $\Omega$ .*

*Proof.* Let  $h_1, h_2$  be two semiconjugacies between  $f$  and  $R_\rho$ . By post-composing with a rigid rotation, we may assume that  $h_1(x) = h_2(x)$  for some  $x \in \Omega$ . Suppose for a contradiction that  $h_1(y) \neq h_2(y)$  for some  $y \in \Omega$ .

Let  $\varepsilon = \frac{1}{2} \cdot d(h_1(y), h_2(y))$  and  $\delta > 0$  such that  $d(h_1(x'), h_2(x')) < \varepsilon$  if  $x' \in B_\delta(x)$  and  $d(h_1(y'), h_2(y')) > \varepsilon$  if  $y' \in B_\delta(y)$ . Due to  $\Omega$  being externally transitive, there exists  $z \in B_\delta(x)$  and  $n \in \mathbb{N}$  such that  $f^n(z) \in B_\delta(y)$ . However, at the same time we have that  $\varepsilon < d(h_1(f^n(z)), h_2(f^n(z))) = d(R_\rho^n(h_1(z)), R_\rho^n(h_2(z))) = d(h_1(z), h_2(z)) < \varepsilon$ , which is absurd.  $\square$

As a consequence, we obtain the uniqueness of the semiconjugacy modulo rotations whenever the non-wandering set of  $f$  is externally transitive. The reason is the following simple observation.

**Lemma 4.2.** *If  $h_1(x) = h_2(x)$  for two semiconjugacies between  $f \in \text{Homeo}(\mathbb{T}^2)$  and a rigid rotation of  $\mathbb{T}^1$ , then  $h_1(y) = h_2(y)$  for all  $y$  with  $x \in \overline{\mathcal{O}(y, f)}$ .*

*Proof.* Suppose for a contradiction that  $x \in \overline{\mathcal{O}(y, f)}$  but  $h_1(y) \neq h_2(y)$ . Let  $\varepsilon = d(h_1(y), h_2(y))/2$  and  $\delta > 0$  such that if  $x' \in B_\delta(x)$  then  $h_1(x'), h_2(x') \in B_\varepsilon(h_1(x))$ . Further, let  $n \in \mathbb{N}$  be such that  $z := f^n(y) \in B_\delta(x)$ . Then on one hand  $h_1(z), h_2(z) \in B_\varepsilon(h_1(x))$ , and on the other hand  $d(h_1(z), h_2(z)) = d(h_1(y), h_2(y)) = 2\varepsilon$ , which is absurd.  $\square$



Given  $f \in \text{Homeo}(\mathbb{T}^2)$  we denote its non-wandering set by  $\Omega(f)$ . Since any orbit accumulates in the non-wandering set, the combination of Proposition 4.1 and Lemma 4.2 immediately yields

**Corollary 4.3.** *Suppose that  $f \in \text{Homeo}(\mathbb{T}^2)$  is semiconjugate to a rigid rotation of  $\mathbb{T}^1$ . Further assume that  $\Omega(f)$  is externally transitive. Then the semiconjugacy is unique modulo rotations.*

For irrational pseudorotations of the torus,<sup>4</sup> external transitivity of the non-wandering set was proved by R. Potrie in [34]. Hence, applying Corollary 4.3 in both coordinates yields

**Corollary 4.4.** *Let  $f \in \text{Homeo}(\mathbb{T}^2)$  be an irrational pseudo-rotation which is semiconjugate to the respective rigid translation of  $\mathbb{T}^2$ . Then the semiconjugacy is unique up to composing with rigid translations of  $\mathbb{T}^2$ .*

Finally, one may ask the following.

**Question 4.5.** Does every semiconjugacy between  $f \in \text{Homeo}_0(\mathbb{T}^2)$  and a rigid rotation on  $\mathbb{T}^1$  have essential annular continua as fibres?

We note that in the example  $f(x, y) = (x + \rho_1, y)$  discussed above this is true, since the fibres of the semiconjugacy  $h_\alpha$  are the essential circles  $\{(x - \alpha(y), y) \mid y \in \mathbb{T}^1\}$ ,  $x \in \mathbb{T}^1$ . By Theorem 1 it is also true whenever the semiconjugacy is unique modulo rotations, since there always exists one semiconjugacy with this property and the topological structure of the fibres is certainly preserved by post-composition with rotations.

## 5. FIBRED ROTATION NUMBER FOR FOLIATIONS OF CIRCLOIDS

The aim of this section is to prove Theorem 2. In order to do so, we need some further preliminary results. Given two open connected subsets  $U, V$  of a manifold  $M$ , we say that  $K \subseteq M \setminus (U \cup V)$  separates  $U$  and  $V$  if  $U$  and  $V$  are contained in different connected components of  $M \setminus K$ .

**Lemma 5.1.** *Suppose  $S \subseteq \mathbb{R}^d$  is a thin horizontal strip and  $K \subseteq S$  is a connected closed set that separates  $\mathcal{U}^+(S)$  and  $\mathcal{U}^-(S)$ . Then  $C_S \subseteq K$ .*

*Proof.* Suppose  $C_S \not\subseteq K$  and let  $z \in C_S \setminus K$ . Then  $B_\varepsilon(z) \subseteq \mathbb{R}^d \setminus K$ . However, as  $B_\varepsilon(z)$  intersects both  $\mathcal{U}^+(S)$  and  $\mathcal{U}^-(S)$  by Lemma 2.2, this means that  $\mathcal{U}^+(S) \cup B_\varepsilon(z) \cup \mathcal{U}^-(S)$  is contained in a single connected component of  $\mathbb{R}^d \setminus K$ , contradicting the fact that  $K$  separates  $\mathcal{U}^+(S)$  and  $\mathcal{U}^-(S)$ .  $\square$

Given an essential thin annular continuum  $A \subseteq \mathbb{A}$ , we denote its lift to  $\mathbb{R}^2$  by  $\hat{A} = \pi^{-1}(A)$ . Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $(x, y) \mapsto (x + 1, y)$ . Then we say  $A$  has a *compact generator*, if there exists a compact connected set  $G_0 \subseteq \hat{A}$  such that  $\bigcup_{n \in \mathbb{Z}} G_n = \hat{A}$ , where  $G_n = T^n(G_0)$ .

**Lemma 5.2.** *If  $A \subseteq \mathbb{A}$  is a thin annular continuum with generator  $G_0$ , then  $G_n \cap G_{n+1} \neq \emptyset$  for all  $n \in \mathbb{N}$ .*

*Proof.* It suffices to prove that  $G_0 \cap G_1 \neq \emptyset$ . As  $C_{\hat{A}} = \bigcup_{n \in \mathbb{Z}} G_n \cap C_{\hat{A}}$  and  $C_{\hat{A}}$  is connected, the compact sets  $G_n \cap C_{\hat{A}}$  cannot be pairwise disjoint. Hence, for some  $k \in \mathbb{N}$  we have  $G_0 \cap G_k \cap C_{\hat{A}} \neq \emptyset$  and we can therefore choose a point  $z_0 \in G_0 \cap C_{\hat{A}}$  with  $z_k = T^k(z_0) \in G_0 \cap G_k$ . If  $k = 1$ , then  $z_1 = T(z_0) \in G_0 \cap G_1$ , and we are finished in this case.

Hence, suppose for a contradiction that  $k > 1$  and  $z_1 = T(z_0) \notin G_0$ . Then  $B_\varepsilon(z_1) \cap G_0 = \emptyset$  for some  $\varepsilon > 0$ . Since  $B_\varepsilon(z_1)$  intersects both  $\mathcal{U}^+(\hat{A})$  and  $\mathcal{U}^-(\hat{A})$  by Lemma 2.2, this means that we can choose a proper curve  $\Gamma \subseteq \mathcal{U}^+(\hat{A}) \cup B_\varepsilon(z_1) \cup \mathcal{U}^-(\hat{A})$  passing through  $z_1$  such that  $\pi_1(\Gamma)$  is bounded and  $\pi_2(\Gamma) = \mathbb{R}$ . In addition, by choosing  $\Gamma$  as the lift of a proper curve  $\gamma \subseteq \mathcal{U}^+(A) \cup B_\varepsilon(\pi(z_1)) \cup \mathcal{U}^-(A) \subseteq \mathbb{A}$  joining  $-\infty$  and  $+\infty$ , we can assume that  $T(\Gamma) \cap \Gamma \neq \emptyset$ .

$\Gamma$  separates  $\mathbb{R}^2$  into two open connected components  $D^-$  unbounded to the left and  $D^+$  unbounded to the right. As  $T(\Gamma)$  is disjoint from  $\Gamma$  we have  $T(\overline{D^+}) \subseteq D^+$ . This implies

<sup>4</sup>That is, torus homeomorphisms homotopic to the identity with unique and totally irrational rotation vector.

that  $z_0 \in D^-$ , otherwise we would have  $z_1 = T(z_0) \in T(\overline{D^+}) \subseteq D^+$ , contradicting  $z_1 \in \Gamma$ . At the same time, we have  $z_k = T^{k-1}(z_1) \in T^{k-1}(\overline{D^+}) \subseteq D^+$ . However, this means that  $z_0$  and  $z_k$  are in different connected components of  $\mathbb{R}^2 \setminus \Gamma$ , contradicting the connectedness of  $G_0$ .  $\square$

Given any bounded set  $B \subseteq \mathbb{R}^2$ , we let

$$\nu_B = \max\{n \in \mathbb{N} \mid \exists z \in B : T^n(z) \in B\}.$$

**Lemma 5.3.** *Suppose  $A, A' \subseteq \mathbb{A}$  are thin essential annular continua with compact generators  $G_0, G'_0$ . Further, assume  $f \in \text{Homeo}_0(\mathbb{A})$  maps  $A$  to  $A'$ . Then for any lift  $F$  of  $f$  the set  $F(G_0)$  intersects at most  $\nu_{G_0} + \nu_{G'_0} + 1$  integer translates of  $G'_0$ .*

*Proof.* Suppose  $F(G_0)$  intersects  $G'_n$  and  $G'_m$  for some  $m > n$ . Then due to Lemma 5.2, the set

$$\bigcup_{k \leq n} G'_k \cup F(G_0) \cup \bigcup_{k \geq m} G'_k \subseteq \widehat{A}$$

is connected and therefore separates  $\mathcal{U}^+(\widehat{A})$  and  $\mathcal{U}^-(\widehat{A})$ . Hence, by Lemma 5.1 it contains  $C_{\widehat{A}} = \widehat{C_A}$ . Let  $z_0 \in G'_0 \cap C_{\widehat{A}}$  and assume without loss of generality that  $z_j \notin G'_0$  for all  $j \geq 1$ . Then  $z_n \in G'_n$  and  $z_j \notin \bigcup_{k \leq n} G'_k$  for all  $j > n$ . Furthermore, since  $z_m \in G'_m$  we have that  $z_j \notin \bigcup_{k \geq m} G'_k$  for all  $j < m - \nu_{G'_0}$ . Thus, we must have

$$\{z_{n+1}, \dots, z_{m-\nu_{G'_0}-1}\} \subseteq F(G_0).$$

However, since  $F(G_0)$  contains at most  $\nu_{G_0}$  integer translates of  $z_0$ , this implies  $m - n \leq \nu_{G_0} + \nu_{G'_0} + 1$ .  $\square$

As a first consequence, this yields the following.

**Corollary 5.4.** *Let  $f \in \text{Homeo}_0(\mathbb{A})$  with lift  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and suppose  $A$  is a thin essential annular continuum which is compactly generated. Then  $f|_A$  has a unique rotation number, that is,*

$$\rho_A(F) = \lim_{n \rightarrow \infty} \pi_2 \circ (F^n(z) - z)/n$$

*exists for all  $z \in \pi^{-1}(A)$  and is independent of  $z$ . Moreover, the convergence is uniform in  $z$ .*

*Proof.* As  $\rho(F, z) = \lim_{n \rightarrow \infty} \pi_2 \circ (F^n(z) - z)/n = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ f^i(z)$  is an ergodic sum with observable  $\varphi(z) = \pi_2(F(z) - z)$ , we have that  $\rho(f, z) = \int_A \varphi \, d\mu =: \rho(\mu)$   $\mu$ -a.s. for every  $f$ -invariant probability measure supported on  $A$ . Note here that  $\varphi$  is well-defined as a function  $\mathbb{A} \rightarrow \mathbb{R}$ . Assume for a contradiction that the rotation number is not unique on  $A$ . Then Theorem 2.5 implies the existence of two  $f$ -invariant ergodic measures  $\mu_1, \mu_2$  supported on  $A$  with  $\rho(\mu_1) \neq \rho(\mu_2)$ . Consequently, we can choose  $z_1, z_2 \in A$  with  $\rho(F, z_1) = \rho(\mu_1) \neq \rho(F, z_2) = \rho(\mu_2)$ . However, at the same time we may choose lifts  $\hat{z}_1, \hat{z}_2 \in G_A$  of  $z_1, z_2$ , where  $G_A$  is a compact generator of  $A$ . Then Lemma 5.3 implies that  $F^n(\hat{z}_1)$  and  $F^n(\hat{z}_2)$  are contained in the union of  $2\nu_{G_A} + 1$  adjacent copies of  $G_A$ . Consequently, we have that  $d(F^n(\hat{z}_1), F^n(\hat{z}_2)) \leq \text{diam}(G_A) + 2\nu_{G_A} + 1$  for all  $n \in \mathbb{N}$ , a contradiction. The uniform convergence follows from the same argument.  $\square$

The following result is a complement of the corollary above.

**Corollary 5.5.** *Let  $f \in \text{Homeo}_0(\mathbb{A})$  with lift  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . Further, suppose  $A$  is a  $f$ -invariant thin essential annular continuum which is compactly generated and  $\rho_A(F) = \{0\}$ . Then  $F$  has a fixed point in  $\pi^{-1}(A)$ .*

*Proof.* Let  $G_0$  be a compact generator of  $A$ , and for every  $k \in \mathbb{Z}$  the set  $G_k := T^k(G_0)$ . We first claim that  $F^n(G_0) \cap G_0 \neq \emptyset$  for all  $n \in \mathbb{Z}$ . Otherwise we have  $n_0 \in \mathbb{N}$  with either  $F^{n_0}(G_0) \subset \bigcup_{k \geq 1} G_k$  or  $F^{n_0}(G_0) \subset \bigcup_{k \leq -1} G_k$ . Let us consider the first situation, for the complementary one a symmetric argument gives the contradiction. Then, for every  $j \in \mathbb{N}$

we have that  $F^{jn_0}(G_0) \subset \bigcup_{k \geq j} G_k$ . This, however, implies that the rotation number of  $A$  is strictly positive, a contradiction.

Consider now  $C := \overline{(\bigcup_{k \in \mathbb{Z}} F^k(G_0))}$ . Thus Lemma 5.3 implies that  $C$  is a compact and invariant set. Moreover, as  $A$  is thin,  $C$  is a non-separating continuum. Then, the Cartwright and Littlewood Theorem [35] implies the existence of a fixed point of  $F$  in  $C$ .  $\square$

**Remark 5.6.** We note that as a special case, Corollary 5.4 applies to decomposable essential thin circloids. In order to see this, recall that a continuum  $C$  is called *decomposable* if it can be written as the union of two non-empty continua  $C_1$  and  $C_2$ . If  $C$  is a thin circloid, then due to the minimality of circloids  $C_1$  and  $C_2$  have to be non-essential. Hence, connected components  $\widehat{C}_i$  of  $\pi^{-1}(C_i) \subseteq \mathbb{R}^2$ ,  $i = 1, 2$ , are bounded. If these lifts are chosen such that their intersection is non-empty, then  $G = \widehat{C}_1 \cup \widehat{C}_2$  is a compact generator of  $C$ .

For this special case, Corollary 5.4 was already proved in [22] by using Caratheodory's Prime End Theory. Examples of (hereditarily) non-decomposable circloids were constructed by Bing [36] and may occur as minimal sets of smooth surface diffeomorphisms [28, 29].

As Lemma 5.3 works for any combination of two compactly generated thin annular continua, we can prove Theorem 2 in a similar way as the above Corollary 5.4. However, what we need as a technical prerequisite is the measurable dependence of the size of the generators of fibres  $h^{-1}(\xi)$  under the assumptions of the theorem. We obtain this in several steps. We place ourselves in the situation of Theorem 2 and assume again without loss of generality that the action  $h^* : \Pi_1(\mathbb{T}^2) \rightarrow \Pi_1(\mathbb{T}^1)$  on the fundamental group is the projection to the second coordinate. This implies that the annular continua  $\mathcal{A}_\xi = h^{-1}\{\xi\}$  are all of homotopy type  $(1, 0)$ . We denote by  $\hat{f}$  the lift of  $f$  to  $\mathbb{A}$  and by  $F$  the lift to  $\mathbb{R}^2$ . Further, we denote by  $\hat{h} : \mathbb{A} \rightarrow \mathbb{R}$  the lift of  $h$  to  $\mathbb{A}$  and by  $H : \mathbb{R}^2 \rightarrow \mathbb{R}$  the lift to  $\mathbb{R}^2$ .

Let  $\Omega_0 = \{\xi \in \mathbb{T}^1 \mid \mathcal{A}_\xi \text{ is thin}\}$ ,  $\Omega = \pi^{-1}(\Omega_0)$  and  $A_\xi = \hat{h}^{-1}\{\xi\}$  ( $\xi \in \mathbb{R}$ ). Then all  $A_\xi$  are essential annular continua in  $\mathbb{A}$ , and  $A_\xi$  is thin if and only if  $\xi \in \Omega$ . Further, define  $A_\xi^+ = \partial \mathcal{U}^+(A_\xi)$  and  $A_\xi^- = \partial \mathcal{U}^-(A_\xi)$ . Then for all  $\xi \in \Omega$  we have  $A_\xi = A_\xi^+ \cup A_\xi^-$  and, by Lemma (2.1),  $A_\xi^+ \cap A_\xi^- = C_{A_\xi} =: C_\xi$ .

We recall that for a metric space  $(X, d)$  and  $C, D \subset X$ , the Hausdorff distance is defined as

$$d_{\mathcal{H}}(C, D) = \max\{\sup_{x \in C} d(x, D), \sup_{y \in D} d(y, C)\}.$$

The convergence of a sequence  $\{C_n\}_{n \in \mathbb{N}}$  of subsets in  $X$  to  $A \subset X$  in this distance is denoted either by  $C_n \rightarrow_{\mathcal{H}} A$  or by  $\lim_{n \rightarrow \infty}^{\mathcal{H}} C_n = A$ . Note that  $d_{\mathcal{H}}(C, D) < \varepsilon$  if and only if  $C \subseteq B_\varepsilon(D)$  and  $D \subseteq B_\varepsilon(C)$ , and that the Hausdorff distance defines a complete metric if one restricts to compact subsets.

**Lemma 5.7.** *If  $A_\xi$  is thin, then  $\lim_{\xi' \nearrow \xi}^{\mathcal{H}} A_{\xi'}^- = \lim_{\xi' \nearrow \xi}^{\mathcal{H}} A_{\xi'} = A_\xi^-$  and  $\lim_{\xi' \searrow \xi}^{\mathcal{H}} A_{\xi'}^+ = \lim_{\xi' \searrow \xi}^{\mathcal{H}} A_{\xi'} = A_\xi^+$ .*

*Proof.* We prove  $\lim_{\xi' \nearrow \xi}^{\mathcal{H}} A_{\xi'}^- = \lim_{\xi' \nearrow \xi}^{\mathcal{H}} A_{\xi'} = A_\xi^-$ , the opposite case follows by symmetry. Since  $A_{\xi_n}^- \subseteq A_{\xi_n}$ , it suffices to show that for all  $\varepsilon > 0$  there exists  $\delta > 0$  such for all  $\xi' \in (\xi - \delta, \xi)$  we have

$$(5.1) \quad A_{\xi'} \subseteq B_\varepsilon(A_\xi^-) \quad \text{and} \quad A_\xi^- \subseteq B_\varepsilon(A_{\xi'}) .$$

We start by showing the first inclusion. Fix  $\varepsilon > 0$ . Assume for a contradiction that there exists a sequence  $\xi_n \nearrow \xi$  such that  $A_{\xi_n} \not\subseteq B_\varepsilon(A_\xi^-)$  for all  $n \in \mathbb{N}$ . Let  $z_n \in A_{\xi_n} \setminus B_\varepsilon(A_\xi^-)$  and  $z = \lim_{n \rightarrow \infty} z_n$ . Then  $z \notin B_\varepsilon(A_\xi^-)$  and thus, since all the  $z_n$  are below  $A_\xi$ , we have  $z \notin A_\xi$ . However, at the same time  $h(z) = \lim_{n \rightarrow \infty} h(z_n) = \lim_{n \rightarrow \infty} \xi_n = \xi$ , a contradiction.

Conversely, in order to show the second inclusion in (5.1), assume for a contradiction that there exists a sequence  $\xi_n \nearrow \xi$  such that  $A_\xi^- \not\subseteq B_\varepsilon(A_{\xi_n}^-)$  for all  $n \in \mathbb{N}$ . Let  $K_n = A_\xi^- \setminus B_\varepsilon(A_{\xi_n}^-)$ . Then  $(K_n)_{n \in \mathbb{N}}$  is a decreasing sequence of non-empty compact sets, such that  $K = \bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$ . Note here that  $A_\xi^- \cap B_\varepsilon(A_{\xi_n}^-) = A_\xi^- \cap B_\varepsilon(\mathcal{U}^-(A_{\xi_n}^-))$ , which is

increasing in  $n$ . Let  $z \in K$ . Then  $B_\varepsilon(z) \cap A_{\xi_n}^- = \emptyset$  and thus  $B_\varepsilon(z) \subseteq \mathcal{U}^+(A_{\xi_n}^-)$  for all  $n \in \mathbb{N}$ . This implies  $h(z') \geq \xi$  for all  $z' \in B_\varepsilon(z)$ , contradicting the fact that  $B_\varepsilon(z)$  intersects  $\mathcal{U}^-(A_\xi)$  and  $h < \xi$  on  $\mathcal{U}^-(A_\xi)$ .  $\square$

Given a compactly generated thin annular continuum  $A$ , we let

$$\tau(A) = \inf\{\text{diam}(G) \mid G \text{ is a compact generator of } A\}.$$

**Lemma 5.8.**  $\xi \mapsto \tau(A_\xi^-)$  is lower semi-continuous from the left on  $\Omega$ , that is,

$$\liminf_{\xi' \nearrow \xi} \tau(A_{\xi'}^-) \geq \tau(A_\xi^-) \quad \text{for all } \xi \in \Omega.$$

Similarly,  $\xi \mapsto \tau(A_\xi^+)$  is lower semi-continuous from the right on  $\Omega$ .

*Proof.* Let  $\xi_n \nearrow \xi$  and assume without loss of generality that  $\tau := \lim_{n \rightarrow \infty} \tau(A_{\xi_n}^-)$  exists and is finite. Choose generators  $G_{\xi_n}$  of  $A_{\xi_n}^-$  of diameter smaller than  $\tau(A_{\xi_n}^-) + \frac{1}{n}$ . Then, using Lemma 5.7, it is straightforward to verify that any limit point  $G$  of  $(G_{\xi_n})_{n \in \mathbb{N}}$  in the Hausdorff metric is a compact generator of  $A_\xi^-$  of diameter smaller than  $\tau$ .  $\square$

It is easy to check that real-valued functions which are lower semi-continuous from one side are also measurable. Consequently, since  $\tau(A_\xi) \leq \eta(\xi) := \tau(A_\xi^-) + \tau(A_\xi^+)$ , the function  $\eta$  provides a measurable majorant for the minimal diameter of the generators of  $A_\xi$ . Further,  $A_\xi^\pm$  are compactly generated if and only if  $A_\xi$  is compactly generated, a fact which follows from the topological considerations on thin annular continua exposed in the next section, see Lemma 6.8 (iv). Altogether, this yields

**Corollary 5.9.** Assume that  $A_\xi$  has compact generator for almost every  $\xi \in \mathbb{T}^1$ . Then the map  $\xi \mapsto \tau(A_\xi)$  has a measurable finite-valued majorant.

We are now in the position to complete the

**Proof of Theorem 2.** Let  $f \in \text{Homeo}_0(\mathbb{T}^2)$  and suppose  $h : \mathbb{T}^2 \rightarrow \mathbb{T}^1$  is a semiconjugacy to the irrational rotation  $R_\rho$ . We assume again that  $h^* = \pi_2^*$ , such that there exist a continuous lift  $\hat{h} : \mathbb{A} \rightarrow \mathbb{R}$  of  $h$  and a lift  $\hat{f} : \mathbb{A} \rightarrow \mathbb{A}$  of  $f$  which satisfy

$$\hat{h} \circ \hat{f} = R_\rho \circ \hat{h}.$$

Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a lift of  $f$ . Assume for a contradiction that  $f$  has no unique rotation number. As in the proof of Corollary 5.4, this implies the existence of two  $f$ -invariant ergodic probability measures  $\mu_1$  and  $\mu_2$  with

$$\rho_1 = \int_{\mathbb{T}^2} \pi_2(F(z) - z) d\mu_1(z) \neq \int_{\mathbb{T}^2} \pi_2(F(z) - z) d\mu_2(z) = \rho_2.$$

As  $h^{-1}\{\xi\}$  is compactly generated for Lebesgue-a.e.  $\xi \in \mathbb{T}^1$ , Corollary 5.9 yields the existence of a finite-valued measurable majorant of  $\xi \mapsto \tau(h^{-1}\{\xi\})$ . Hence, we can find a constant  $C > 0$  and a set  $\Omega_C \subseteq \mathbb{T}^1$  of positive measure such that for all  $\xi \in \Omega_C$  the annular continuum  $h^{-1}\{\xi\}$  has a compact generator  $G_\xi$  with  $\text{diam}(G_\xi) \leq C$ .

Both  $\mu_1$  and  $\mu_2$  must be mapped to the Lebesgue measure on  $\mathbb{T}^1$  by  $h$ , since this is the only invariant probability measure of  $R_\rho$ . Hence, for almost every  $\xi \in \mathbb{T}^1$  there exist points  $z_1, z_2 \in h^{-1}\{\xi\}$  which are generic with respect to  $\mu_1$  and  $\mu_2$ , respectively. In particular, for any lift  $\hat{z}_i \in \mathbb{R}^2$  of  $z_i$  we have that

$$(5.2) \quad \lim_{n \rightarrow \infty} \pi_2(F^n(\hat{z}_i) - \hat{z}_i)/n = \rho_i \quad (i = 1, 2).$$

Without loss of generality, we may assume that  $h^{-1}\{\xi\}$  has compact generator  $G_\xi$  and  $R_\rho^n(\xi)$  visits  $\Omega_C$  infinitely many times, say,  $r_\rho^{n_i}(\xi) \in \Omega_C$  for a strictly increasing sequence  $(n_i)_{i \in \mathbb{N}}$  of integers. Given lifts  $\hat{z}_1, \hat{z}_2 \in G_\xi$  of  $z_1, z_2$ , Lemma 5.3 implies that

$$\pi_2(F^{n_i}(\hat{z}_1)) - \pi_2(F^{n_i}(\hat{z}_2)) \leq \text{diam}(G_{r_\rho^{n_i}(\xi)}) + \nu_{G_\xi} + \nu_{G_{r_\rho^{n_i}(\xi)}} + 1 \leq \nu_{G_\xi} + 2C + 1$$

for all  $i \in \mathbb{N}$ . As  $\rho_1 \neq \rho_2$ , this contradicts (5.2).  $\square$

## 6. COMMENTS ON THE TOPOLOGY AND ROTATION SETS OF THIN ANNULAR CONTINUA.

Given  $X \subset \mathbb{R}^2$  we denote by  $[X]_y$  the connected component of  $y$  in  $X$  and define  $h(X) = \sup\{x_1 - x_2 \mid (x_1, y_1), (x_2, y_2) \in X\}$ . For an essential annular continuum  $A \subset \mathbb{A}$ , we define the set of *spikes* of  $A$  as

$$\mathcal{S}_A := \{[(\hat{A} \setminus C_{\hat{A}})]_x \mid x \in \hat{A} \setminus C_{\hat{A}}\}$$

and say that  $A$  has an infinite spike if there exists  $S \in \mathcal{S}_A$  with  $h(S) = \infty$ . Further we let  $H_{\mathcal{S}_A} := \sup\{h(S) \mid S \in \mathcal{S}_A\}$ . The main result of this section is the following.

**Theorem 6.1.** *Let  $A \subset \mathbb{A}$  be an essential thin annular continuum. Then the following holds.*

- (1) *If  $A$  is not compactly generated then either*
  - (1a)  *$C_A$  is not compactly generated, or*
  - (1b)  *$C_A$  is compactly generated and  $A$  contains an infinite spike.*
- (2) *If  $A$  compactly generated, then so is  $C_A$ .*

The proof is given in Section 6.1 below. As an example in the class of continua given in (1a), we can regard the *Birkhoff attractor*. This is an essential thin circloid which has a segment as a rotation set for some map that leaves it invariant (see e.g. [26]). Hence, due to Corollary 5.4 the Birkhoff attractor cannot have a compact generator. For the class given in (1b) we can consider the continuum given by  $A = \pi(\mathbb{R} \times \{0\} \cup \{(x, \frac{1}{x}) \in \mathbb{R}^2 \mid x \geq 1\})$ , which contains the infinite spike  $S = \{(x, \frac{1}{x}) \in \mathbb{R}^2 \mid x \geq 1\}$ .

Note that Theorem 6.1 does not rule out the coexistence of an infinite spike and a compact generator. In fact, this may happen, and a way to construct such examples is the following. Let  $I = [0, 1] \times \{0\}$  and  $J = \{0\} \times [0, 1]$ . We consider  $K = J \cup I \cup T(J)$ . Fix  $x_0 \in J \setminus I$  and  $x_1 = T(x_0)$  and let  $\gamma : [0, +\infty) \rightarrow \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, y > 0\}$  be an injective curve that verifies

- (i)  $\gamma([n, +\infty)) \subset B_{\frac{1}{n}}(K)$  for every  $n \in \mathbb{N}$ ;
- (ii)  $\lim_i \gamma(t_i) = x_0$ ,  $\lim_j \gamma(t_j) = x_1$  for two strictly increasing sequences of positive integers  $(t_i)_{i \in \mathbb{N}}, (t_j)_{j \in \mathbb{N}}$ .

Now let  $A = \pi(\hat{A})$  where  $\hat{A} := \bigcup_{n \in \mathbb{Z}} T^n(K \cup \gamma)$ . It is easy to see that  $A$  is a thin essential annular continuum. Furthermore the set  $G = K \cup \gamma$  is compact and connected, and hence a generator of  $A$ . Finally the set  $S := \hat{A} \setminus (\mathbb{R} \times \{0\})$  is connected since  $S = \bigcup_{n \in \mathbb{Z}} T^n((J \cup T(J) \setminus I) \cup \gamma)$ . Hence  $A$  has compact generator  $G$  and at the same time contains the infinite spike  $S$ . What is not clear to us is whether similar examples can be produced with an infinite spike that is not  $T$ -invariant.

**Question 6.2.** Suppose  $A$  is a thin annular continuum which contains an infinite spike  $S$  with  $T^n(S) \cap S = \emptyset$  for all  $n \in \mathbb{N}$ . Does this imply that  $A$  has no compact generator?

As seen in Corollary 5.4, an invariant compactly generated thin annular continuum has a unique rotation vector. As the Birkhoff attractor shows, this is not true if  $C_A$  has no generator. For the case of thin annular continua which are not compactly generated, but have a compactly generated circloid, the situation is similar.

**Proposition 6.3.** *Given any segment  $I \subset \mathbb{R}$ , there exists a map  $f \in \text{Homeo}(A)$  which leaves invariant an essential thin annular continuum  $A \subset \mathbb{A}$  such that  $C_A$  has compact generator,  $A$  has an infinite spike, and  $\rho_A(f) = I$ .*

The proof, which is similar to a construction by Walker [30] of some examples in the context of prime end rotation, is given in Section 6.2.

**6.1. Topology of thin annular continua: Proof of Theorem 6.1.** The proof of Theorem 6.1 mainly hinges on a number of purely plane-topological lemmas. In order to state and prove them, we need to introduce further notation. Given a Jordan curve  $\gamma \subset \mathbb{R}^2$  we denote its Jordan domain by  $D_\gamma$ . Further, for two curves  $\alpha, \beta$  in the plane with  $\alpha(0) = \beta(1)$ , we denote their concatenation by  $\alpha \cdot \beta$ . We say an arc is an injective curve  $\alpha : [0, 1] \rightarrow \mathbb{R}^2$  and define  $\hat{\alpha} := \alpha \setminus \{\alpha(0), \alpha(1)\}$ . We will sometimes abuse terminology by identifying the function

defining a curve and its image. The diameter of a set  $X \subset \mathbb{R}^2$  is denoted by  $\text{diam}(X)$ . As before, given two sets  $A, B \subset \mathbb{R}^2$  we denote the Hausdorff distance by  $d_{\mathcal{H}}(A, B)$ . Our first step is to prove that the existence of a generator for an essential thin annular continuum  $A$  implies the existence of a generator for the circloid  $C_A$ . We divide the proof into several lemmas.

**Lemma 6.4.** *Given a bounded set  $X \subset \mathbb{R}^2$  and two positive numbers  $\delta_1 < \delta_2$ , there exists a neighbourhood  $U_{\delta_1\delta_2}(X)$  of  $X$  which verifies*

- (i)  $B_{\delta_1}(X) \subset U_{\delta_1\delta_2}(X) \subset B_{\delta_2}(X)$ ;
- (ii)  $U_{\delta_1\delta_2}(X) = \bigcup_{i=1}^n D_i$  where  $\{\overline{D_i}\}_{i=1}^n$  is a family of pairwise disjoint topological closed disks with  $C^1$ -boundaries  $\partial D_i$ .

*Proof.* Let  $\Delta : \mathbb{R}^2 \rightarrow \mathbb{R}$  be given by  $\Delta(y) = d(y, \overline{X})$ . Let  $\varepsilon_1 = \min\{\frac{\delta_1}{10}, \frac{\delta_2 - \delta_1}{10}\}$ , such that  $a := \delta_1 + \varepsilon_1 < \delta_2 + \varepsilon_2 =: b$ . Further, let  $\Psi : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a  $C^\infty$  function such that  $d_0(\Psi, \Delta) < \varepsilon_1$ . Then  $\Psi^{-1}((-\infty, a)) \supset B_{\delta_1}(\overline{X})$  and  $\Psi^{-1}((-\infty, b)) \subset B_{\delta_2}(\overline{X})$ .

Consider now a regular value<sup>5</sup>  $r \in (a, b)$  of  $D$  which exists due to Sard's Theorem [37, Chapter 1, Section 7]. Then we have that  $\Psi^{-1}(r)$  is a union of pairwise disjoint  $C^1$  Jordan curves  $\{C_j\}_{j=1}^m$ . Hence,  $U_{\delta_1\delta_2}(X) := \Psi^{-1}((-\infty, r))$  has all the desired properties. Note here that  $X \subseteq \Psi^{-1}((-\infty, r))$  and  $\partial\Psi^{-1}((-\infty, r)) \subseteq \Psi^{-1}(\{r\})$ .  $\square$

**Lemma 6.5.** *Let  $B_1, B_2$  be two open sets in  $\mathbb{R}^2$  with  $d(B_1, B_2) = \varepsilon_0 > 0$  and fix  $\varepsilon_1 > 0$ . Then there exists  $\xi = \xi(\varepsilon_0, \varepsilon_1) > 0$  such that given any pair of arcs  $\alpha$  and  $\beta$  which verify*

- (i)  $\alpha(0) = \beta(1) \in B_1$ ,  $\alpha(1) = \beta(0) \in B_2$  and  $\alpha \cdot \beta$  is a Jordan curve,
- (ii)  $d(\alpha \setminus (\overline{B_1 \cup B_2}), \beta \setminus (\overline{B_1 \cup B_2})) \geq \varepsilon_1$ ,

*there exists  $z \in D_{\alpha\cdot\beta}$  for which  $B_\xi(z) \subset D_{\alpha\cdot\beta} \setminus (\overline{B_1 \cup B_2})$ . (See Figure 6.1).*

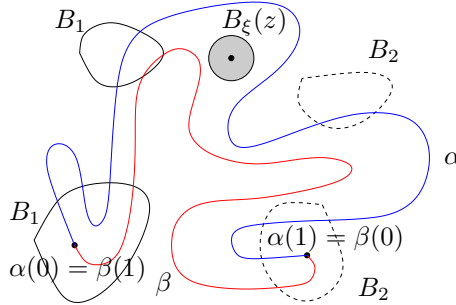


FIGURE 6.1.

*Proof.* Let  $\delta_1 = \frac{\varepsilon_0}{10}$  and  $\delta_2 = \frac{\varepsilon_0}{5}$ . We consider  $U_1 := U_{\delta_1\delta_2}(B_1)$  and  $U_2 := U_{\delta_1\delta_2}(B_2)$  given by the previous lemma. Due to the fact that continuous arcs can be approximated by  $C^1$  arcs, we may assume that the arcs  $\alpha, \beta$  are  $C^1$ . Furthermore, for similar reasons we may assume that the curves  $\alpha, \beta$  are transverse to each Jordan curve in  $\partial U_1 \cup \partial U_2$ .

Due to  $U_1$  being disjoint of  $U_2$ , we have a  $C^1$  loop  $\theta \subset \partial U_1$  that separates  $\alpha(0)$  from  $\alpha(1)$ . We claim that there exists a subarc  $J \subseteq \theta$  such that

- (a)  $J(0) \in \alpha \cap \theta$  and  $J(1) \in \beta \cap \theta$ ;
- (b)  $\overset{\circ}{J} \subset D_{\alpha\cdot\beta}$ .

In order to see this, fix a point  $x_0 \in \theta$  outside of  $\overline{D_{\alpha\cdot\beta}}$  and assume, by reparametrising if necessary, that  $\theta(0) = x_0$ . Then, if  $J$  with the above properties does not exist, this means that every time that  $\theta$  goes into  $D_{\alpha\cdot\beta}$  by passing through  $\alpha$ , it goes out of  $D_{\alpha\cdot\beta}$  through  $\alpha$  as well. Consequently, the algebraic intersection number  $[\alpha] \wedge [\theta]$  between the homotopy

<sup>5</sup>That is, a value  $r$  such that for all  $y \in \Psi^{-1}(r)$  the derivative matrix  $D\Psi(y)$  is surjective.

classes  $[\alpha], [\theta]$  relative to the points  $x_0, \alpha(0), \alpha(1)$ , is zero. However, this is a contradiction since  $\theta$  separates  $\alpha(0)$  and  $\alpha(1)$ , so  $[\alpha] \wedge [\theta] = 1$  (compare [37, Chapter 3]).

Therefore, we can consider an arc  $J$  with the above properties. Let  $\xi := \min\{\frac{\varepsilon_0}{10}, \frac{\varepsilon_1}{2}\}$  and consider a bijective parametrisation  $\gamma : [0, 1] \rightarrow J$  with  $\gamma(0) = z_0 \in \alpha$  and  $\gamma(1) = z_1 \in \beta$  (see Figure 6.2).

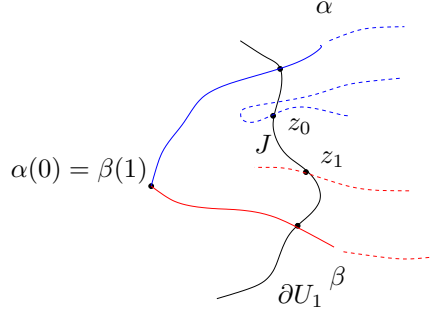


FIGURE 6.2.

Let  $\lambda := \sup\{t \in [0, 1] \mid d(\gamma(t), \alpha) < \xi\}$  which exists since 1 is an upper bound of  $\{t \in [0, 1] \mid d(\gamma(t), \alpha) < \xi\}$ . Otherwise we would have that  $d(\alpha, \beta) < \varepsilon_1$  in  $(B_1 \cup B_2)^c$ .

Then, we have for  $z := \gamma(\lambda)$  that  $B_\xi(z) \cap \alpha = \emptyset$  and  $B_{\xi+\delta}(z) \cap \alpha \neq \emptyset$  for any positive number  $\delta$ . Thus we have  $B_\xi(z) \cap \beta = \emptyset$ , otherwise  $d(\alpha, \beta) < \varepsilon_1$  in  $(\overline{B_1 \cup B_2})^c$  which is impossible. This implies that  $B_\xi(z) \subset D_{\alpha, \beta} \setminus (\overline{B_1 \cup B_2})$ .  $\square$

**Lemma 6.6.** *Suppose  $(\alpha_n)_{n \in \mathbb{N}}$  and  $(\beta_n)_{n \in \mathbb{N}}$  are two sequences of arcs in  $\mathbb{R}^2$  with the following properties:*

- (i)  $\alpha_n, \beta_n \subset B_R(0)$  and  $\alpha_n \cap \beta_n = \emptyset$  for every  $n \in \mathbb{N}$  and some  $R \in \mathbb{R}$ ;
- (ii)  $\lim_{n \rightarrow \infty}^{\mathcal{H}} \alpha_n = \mathcal{L}_1, \lim_{n \rightarrow \infty}^{\mathcal{H}} \beta_n = \mathcal{L}_2$ ;
- (iii)  $(\alpha_n(0))_{n \in \mathbb{N}}$  and  $(\beta_n(0))_{n \in \mathbb{N}}$  converge to  $x_0 \in \mathbb{R}^2$ ,  $(\alpha_n(1))_{n \in \mathbb{N}}$  and  $(\beta_n(1))_{n \in \mathbb{N}}$  converges to  $x_1 \in \mathbb{R}^2$  with  $x_0 \neq x_1$ .

Then we have that either  $[\mathcal{L}_1 \cap \mathcal{L}_2]_{x_0} = [\mathcal{L}_1 \cap \mathcal{L}_2]_{x_1}$  or  $\partial(\mathcal{L}_1 \cup \mathcal{L}_2)$  separates  $\mathbb{R}^2$ .

*Proof.* Suppose  $[\mathcal{L}_1 \cap \mathcal{L}_2]_{x_0} \neq [\mathcal{L}_1 \cap \mathcal{L}_2]_{x_1}$ . We claim that for some  $\delta > 0$  there exist two different connected components  $B_1, B_2$  of  $B_\delta(\mathcal{L}_1 \cup \mathcal{L}_2)$ , such that

- $d(\overline{B_1}, \overline{B_2}) = \varepsilon_0 > 0$ ;
- if  $C_1 = B_1 \cap (\mathcal{L}_1 \cap \mathcal{L}_2)$  and  $C_2 = B_2 \cap (\mathcal{L}_1 \cap \mathcal{L}_2)$  then we have  $x_0 \in C_1, x_1 \in C_2$  and  $\mathcal{L}_1 \cap \mathcal{L}_2 = C_1 \cup C_2$ .

Otherwise for every  $\varepsilon > 0$  the points  $x_0$  and  $x_1$  would be in the same connected component of  $B_\varepsilon(\mathcal{L}_1 \cap \mathcal{L}_2)$ . However, this would imply that  $K := \bigcap_{n \in \mathbb{N}} [B_{\frac{1}{n}}(\mathcal{L}_1 \cap \mathcal{L}_2)]_{x_0} \subset \mathcal{L}_1 \cap \mathcal{L}_2$  is a connected set containing  $x_0$  and  $x_1$ , contradicting  $[\mathcal{L}_1 \cap \mathcal{L}_2]_{x_0} \neq [\mathcal{L}_1 \cap \mathcal{L}_2]_{x_1}$ .

Hence, we can choose  $B_1, B_2$  as above. Then, for every  $n \in \mathbb{N}$  we consider small arcs  $a_n, b_n, c_n$  and  $d_n$  such that  $a_n$  joins  $\alpha_n(1)$  with  $x_1$ ,  $b_n$  joins  $x_1$  with  $\beta_n(1)$ ,  $c_n$  joins  $\beta_n(0)$  with  $x_0$  and  $d_n$  joins  $x_0$  with  $\alpha_n(0)$ . Further we may request that  $\gamma_n := \alpha_n \cdot a_n \cdot b_n \cdot \beta_n^{-1} \cdot c_n \cdot d_n$  is a  $C^1$  Jordan curve, and that  $\lim_{n \rightarrow \infty}^{\mathcal{H}} a_n \cdot b_n = x_1, \lim_{n \rightarrow \infty}^{\mathcal{H}} c_n \cdot d_n = x_0$  (See Figure 6.3).

Let  $\omega_n := d_n \cdot \alpha_n \cdot a_n$  and  $\theta_n := b_n \cdot \beta_n \cdot c_n$ ,  $D_n := D_{\omega_n \cdot \theta_n} = D_{\gamma_n}$ . Since  $\mathcal{L}_1 \cap \mathcal{L}_2$  is contained in  $B_1 \cup B_2$  and we have  $\lim_{n \rightarrow \infty}^{\mathcal{H}} \omega_n = \mathcal{L}_1$  and  $\lim_{n \rightarrow \infty}^{\mathcal{H}} \theta_n = \mathcal{L}_2$ , there exists  $\varepsilon_1 > 0$  such that for every  $n \in \mathbb{N}$

$$d(\omega_n \setminus (\overline{B_1 \cup B_2}), \theta_n \setminus (\overline{B_1 \cup B_2})) \geq \varepsilon_1$$

Consequently, due to Lemma 6.5 there exists a positive number  $\xi$  and a sequence of points  $(z_n)_{n \in \mathbb{N}}$  such that  $B_\xi(z_n) \subset D_n \setminus (\overline{B_1 \cup B_2})$  for every  $n \in \mathbb{N}$ . Therefore, if we consider a subsequence  $(D_{n_i})_{i \in \mathbb{N}}$  which converges to  $D_0$ , we have the existence of some  $z_0 \in D_0$  such that  $B_\xi(z_0) \subset D_0 \setminus (\overline{B_1 \cup B_2})$ . By construction, every curve joining  $z \in B_\xi(z_0)$

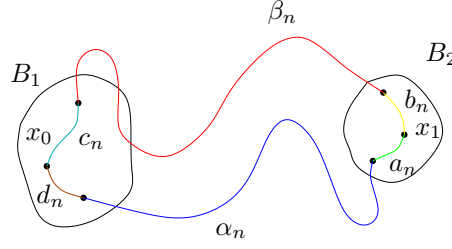


FIGURE 6.3.

with  $\infty$  intersects  $\partial D_0 \subset \partial(\mathcal{L}_1 \cup \mathcal{L}_2)$ . Hence  $\partial(\mathcal{L}_1 \cup \mathcal{L}_2)$  separates  $\mathbb{R}^2$  into at least two components.  $\square$

**Lemma 6.7.** *Let  $D \subset \mathbb{R}^2$  be a smooth disk such that  $\text{int}(D) \cap \text{int}(T(D)) \neq \emptyset$ . Further consider the embedding of the real line  $\Gamma_1 = \partial\mathcal{U}^-(\bigcup_{n \in \mathbb{Z}} T^n(\overline{D}))$ . Then for every  $x \in \Gamma_1$  and every arc  $\gamma_x$  with endpoints  $x, T(x)$ , we have  $h(\gamma_x) < 4 \cdot \text{diam}(D) + 4$ . The analogous statement holds for  $\Gamma_2 = \partial\mathcal{U}^+(\bigcup_{n \in \mathbb{Z}} T^n(\overline{D}))$ .*

*Proof.* We first fix  $x \in \Gamma_1$  with the property that  $\pi_2(x) = \max \pi_2(\Gamma_1)$ . Let  $D_k := T^k(D)$ . Suppose for a contradiction that  $h(\gamma_x) \geq 2 \cdot \text{diam}(D) + 2$ . Let  $z \in \gamma_x$  maximise the function  $|\pi_1(\cdot) - \pi_1(x)| : \gamma_x \rightarrow \mathbb{R}$ . Then  $|\pi_1(z) - \pi_1(x)| \geq \text{diam}(D) + 1$ . Further, due to the  $T$ -invariance of  $\Gamma_1$  and the choice of  $x$  we have that  $\gamma_x$  verifies  $\gamma_x \cap (\mathbb{R} \times \{\pi_2(x)\}) = \gamma_x \cap s_x$ , where  $s_x$  is the interval joining  $x$  with  $T(x)$ .

Let  $\Delta_x \subset \gamma_x$  be an arc containing  $z$ , such that  $\Delta_x(0) = x_0, \Delta_x(1) = x_1 \in s_x$  and  $\Delta_x(t) \notin s_x$  for all  $t \in (0, 1)$ . Further, let  $I_x$  be the segment joining  $x_1$  with  $x_0$  and  $\alpha_x := \Delta_x \cdot I_x$ . Then  $D_{\alpha_x} \subset \mathcal{U}^+(\Gamma_1)$  since  $\overset{\circ}{I_x} \subset \overset{\circ}{\mathcal{U}^+(\Gamma_1)}$ . This implies for  $D_k$  with  $z \in \partial D_k$  that  $D_k \subset D_{\alpha_x}$ . Otherwise we would have  $D_k \cap \overset{\circ}{I_x} \neq \emptyset$  which implies that  $\text{diam}(D_k) > \text{diam}(D)$ . On the other hand, we have by construction that  $T(D_{\alpha_x}) \cap D_{\alpha_x} = \emptyset$ . Hence  $T(D_k) \cap D_k = \emptyset$ , which is absurd.

Given now any point  $y \in \Gamma_1$  we find  $v = (n, 0)$  with  $n \in \mathbb{Z}$  such that  $y \in \gamma_{x+v}$ . Thus  $T(y) \in \gamma_{x+v+(1,0)}$ , and  $\gamma_y$  is an arc contained in  $\gamma_{x+v} \cup \gamma_{x+v+(1,0)}$ . This implies that  $h(\gamma_y) < 2 \cdot h(\gamma_x) < 4 \cdot \text{diam}(D) + 4$ .  $\square$

**Lemma 6.8.** *Let  $A \subset \mathbb{A}$  be an essential thin annular continuum with compact generator. Then, there exist two sequences of arcs  $(\alpha_n)_{n \in \mathbb{N}}$  and  $(\beta_n)_{n \in \mathbb{N}}$  which verify:*

- (i)  $\alpha_n \subset \mathcal{U}^-(\hat{A})$  and  $\beta_n \subset \mathcal{U}^+(\hat{A})$  for every  $n \in \mathbb{N}$ ;
- (ii)  $(\alpha_n(0))_{n \in \mathbb{N}}, (\beta_n(0))_{n \in \mathbb{N}}$  converges to  $x_0 \in C_{\hat{A}}$  and  $(\alpha_n(1))_{n \in \mathbb{N}}, (\beta_n(1))_{n \in \mathbb{N}}$  converges to  $x_1 = T(x_0)$ ;
- (iii)  $\lim_{n \rightarrow \infty}^{\mathcal{H}} \alpha_n = \mathcal{L}_1$  and  $\lim_{n \rightarrow \infty}^{\mathcal{H}} \beta_n = \mathcal{L}_2$  exist;
- (iv)  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are generators of  $A^- = \partial\mathcal{U}^-(A)$  and  $A^+ = \partial\mathcal{U}^+(A)$ , respectively, and  $G := \mathcal{L}_1 \cup \mathcal{L}_2$  is a generator of  $A$ .

*Proof.* Let  $G'$  be a generator of  $A$ . Due to the Riemann Mapping Theorem applied in  $\mathbb{R}^2 \cup \{\infty\} \setminus G'$ , we can consider a sequence of smooth Jordan curves  $(\gamma_n)_{n \in \mathbb{N}}$  such that

- (1)  $\gamma_n \subset \mathbb{R}^2 \setminus G'$  for every  $n \in \mathbb{N}$ ;
- (2)  $\lim_{n \rightarrow \infty}^{\mathcal{H}} \gamma_n = G'$ ;
- (3)  $D_{n+1} \subset D_n$ , where  $D_n = D_{\gamma_n}$  for every  $n \in \mathbb{N}$ .

Due to (2) we have  $\text{diam}(D_n) < K_0$  for some  $K_0 \in \mathbb{R}$  and all  $n \in \mathbb{N}$ . For every  $n \in \mathbb{N}$ ,  $m \in \mathbb{Z}$  we define  $D_{n,m} = T^m(D_n)$ ,  $\mathcal{D}_n = \bigcup_{m \in \mathbb{Z}} D_{n,m}$ , and  $\Gamma_n^- = \partial\mathcal{U}^-(\mathcal{D}_n)$ ,  $\Gamma_n^+ = \partial\mathcal{U}^+(\mathcal{D}_n)$ . Notice that the curves  $\Gamma_n^-, \Gamma_n^+$  are embeddings of the real line of the same form as in the previous lemma. Let  $x_0 \in G' \cap C_{\hat{A}}$ .

Given  $n \in \mathbb{N}$  we consider two arcs  $\alpha'_n \subset \Gamma_n^-$  and  $\beta'_n \subset \Gamma_n^+$  such that  $\alpha'_n(1) = T(\alpha'_n(0))$ ,  $\beta'_n(1) = T(\beta'_n(0))$  and  $(\alpha'_n(0))_{n \in \mathbb{N}}, (\beta'_n(0))_{n \in \mathbb{N}}$  converge to  $x_0$ . Due to Lemma 6.7 we know



that  $\alpha'_n, \beta'_n$  are contained in a ball  $B_R(x_0)$  where

$$R = 8(K_0 + 1) + \max\{d(\alpha'_n(0), \beta'_n(0)) \mid n \in \mathbb{N}\}.$$

Then by choosing subsequences of  $(\alpha'_n)_{n \in \mathbb{N}}, (\beta'_n)_{n \in \mathbb{N}}$  we obtain two sequences  $(\alpha_n)_{n \in \mathbb{N}}, (\beta_n)_{n \in \mathbb{N}}$  that verify statements (i) and (ii) of the theorem and the limits  $\mathcal{L}_1$  and  $\mathcal{L}_2$  exist. This implies in particular that  $G = \mathcal{L}_1 \cup \mathcal{L}_2$  is a subcontinuum of  $\hat{A}$ . Moreover, as  $\alpha_n, \beta_n$  are generators of  $\Gamma_{k_n}^-, \Gamma_{k_n}^+$  for a suitable sequence  $(k_n)_{n \in \mathbb{N}}$  and  $\left(\lim_{n \rightarrow \infty}^{\mathcal{H}} \Gamma_{k_n}^- \cup \lim_{n \rightarrow \infty}^{\mathcal{H}} \Gamma_{k_n}^+\right) = \hat{A}$  since  $A$  is thin, we have that  $\mathcal{L}_1$  is a generator of  $\partial \mathcal{U}^-(A)$ ,  $\mathcal{L}_2$  is a generator of  $\partial \mathcal{U}^+(A)$  and  $G$  is a generator of  $A$ .  $\square$

As consequence of the above results, we obtain the following.

**Corollary 6.9.** *If an essential thin annular continuum  $A$  has generator, then the circloid  $C_A$  has generator.*

*Proof.* Let  $G = \partial(\mathcal{L}_1 \cup \mathcal{L}_2)$  be the generator of  $A$  given by Lemma 6.8. Due to Lemma 6.6 we have  $G' := [\mathcal{L}_1 \cap \mathcal{L}_2]_{x_0} = [\mathcal{L}_1 \cap \mathcal{L}_2]_{T(x_0)}$ . Further, since  $\mathcal{L}_1 \subseteq \partial \mathcal{U}^+(A)$  and  $\mathcal{L}_2 \subseteq \partial \mathcal{U}^-(A)$  we have that  $\mathcal{L}_1 \cap \mathcal{L}_2 \subseteq C_A$  by Lemma 2.2. This implies that  $C := \pi\left(\bigcup_{n \in \mathbb{N}} T^n(G')\right)$  is an essential annular continuum contained  $C_A$ , and hence  $C = C_A$  since  $C_A$  is a circloid. Thus  $G'$  is a generator for  $C_A$ .  $\square$

We now consider the family of all the essential thin annular continua  $\mathcal{A}$ , and let  $\mathcal{A}_1$  be the family of those  $A \in \mathcal{A}$  such that  $C_A$  has no generator and  $\mathcal{A}_2 = \mathcal{A} \setminus \mathcal{A}_1$ . Then due to the last corollary  $\mathcal{A}_2$  contains those  $A \in \mathcal{A}$  which admit compact generator. In order to finish the proof of Theorem 6.1, it remains to show that whenever  $A \in \mathcal{A}_2$  has no generator then  $A$  contains an infinite spike. We proceed in two steps and start by showing that every finite spike has at least one ‘base point’ in  $C_A$ .

**Lemma 6.10.** *If  $A$  is a thin annular continuum and  $S$  is a spike of  $A$  with  $h(S) < \infty$ , then  $\overline{S} \cap C_{\hat{A}} \neq \emptyset$ .*

*Proof.* Fix  $x \in S$ . For each  $n \in \mathbb{N}$  we define  $\mathcal{Y}_n$  as the family of all curves  $\alpha : [0, 1] \rightarrow \mathbb{R}^2$  which verify:

- (i)  $\alpha(0) \in B_{\frac{1}{n}}(x)$ ;
- (ii)  $\alpha(1) \in B_{\frac{1}{n}}(C_{\hat{A}})$ ;
- (iii)  $\alpha \subset B_{\frac{1}{n}}(\hat{A})$ .

Let  $\mathbb{R}^2 \cup \{\infty\}$  be the compactification of  $\mathbb{R}^2$  by the sphere. Then, we can consider a sequence of curves  $(\alpha_n)_{n \in \mathbb{N}}$  with  $\alpha_n \in \mathcal{Y}_{k_n}$  such that:

- $k_n \nearrow \infty$ ;
- $\lim_{n \rightarrow \infty}^{\mathcal{H}} \alpha_n = \mathcal{L}$  in  $\mathbb{R}^2 \cup \{\infty\}$ .

Let  $\mathcal{L}_x := [\mathcal{L} \setminus (C_{\hat{A}} \cup \{\infty\})]_x$ . Then  $\mathcal{L}_x \subset S$ , and hence  $\infty \notin \overline{\mathcal{L}_x}$  since  $h(S) < \infty$ , so  $\overline{\mathcal{L}_x}$  is compact. By property (ii) of the curves  $\alpha_n$  we have  $\mathcal{L}_x \cap B_{\frac{1}{n}}(C_{\hat{A}}) \neq \emptyset$  for all  $n \in \mathbb{N}$ , and hence  $\overline{\mathcal{L}_x} \cap C_{\hat{A}} \neq \emptyset$ .  $\square$

**Corollary 6.11.** *If  $A \in \mathcal{A}_2$  and  $H_{S_A} < \infty$ , then  $A$  has generator.*

*Proof.* Let  $G'$  be a generator of  $C_A$ . For every spike  $S$  choose  $n \in \mathbb{N}$  such that  $S' := T^n(S)$  intersects  $G'$ . Note that this is possible due to Lemma 6.10. Since  $H_{S_A} < \infty$  we have that  $G := \overline{(G' \cup \bigcup_{S \in \mathcal{S}_A} S')}$  is a compact generator of  $A$ .  $\square$

Finally, we show that for every  $A \in \mathcal{A}_2$  with  $H_{S_A} = \infty$ , there exists an infinite spike contained in  $A$ .

**Proposition 6.12.** *Let  $A \in \mathcal{A}_2$  with  $H_{S_A} = \infty$ . Then, there exists an infinite spike  $S \in \mathcal{S}_A$ .*

*Proof.* We assume that the supremum  $H_{S_A}$  is obtained by spikes in  $\mathcal{U}^-(C_{\hat{A}})$ , the other case is symmetric. Suppose for a contradiction that  $h(S) < \infty$  for every  $S \in \mathcal{S}_A$ .

Let  $x_0 \in \hat{A} \setminus \mathcal{U}^+(C_{\hat{A}}) = \left( \hat{A} \cap \mathcal{U}^-(C_{\hat{A}}) \right) \cup C_{\hat{A}}$  such that

$$\pi_2(x_0) = \min \left\{ \pi_2(x) \mid x \in \bigcup_{S \in \mathcal{S}_A} S \cap \mathcal{U}^-(C_{\hat{A}}) \right\}.$$

By changing coordinates if necessary, we may assume that  $x_0 \notin C_{\hat{A}}$ .

Let  $\gamma_{x_0}(t) = x_0 + t \cdot (1, 0)$  and  $S_0 \in \mathcal{S}_A$  such that  $x_0 \in S_0$ . Then due to Lemma 6.10 and the fact that  $C_A$  has generator, we can consider a generator  $L$  of  $C_A$  that verifies  $L \cap \overline{S_0} \neq \emptyset$  and  $L \cap \overline{T(S_0)} \neq \emptyset$  (see Figure 6.4).

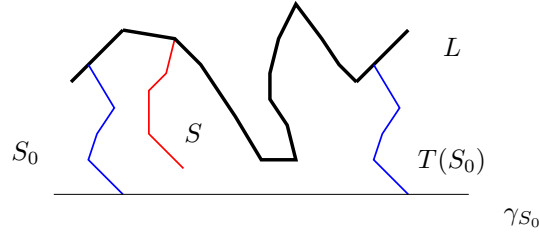


FIGURE 6.4.

Given now any spike  $S \subset \mathcal{U}^-(C_{\hat{A}})$  different of  $S_0$ , due to the definition of  $x_0$  we have:

$$S \subset \left( \overline{\mathcal{U}^+(\gamma_{x_0})} \cap \mathcal{U}^-\left(\bigcup_{n \in \mathbb{Z}} T^n(L)\right) \right) \setminus \bigcup_{n \in \mathbb{Z}} T^n(S_0).$$

Therefore we have that  $h(S) < 2 \cdot h(S_0) + h(L)$ . This contradicts  $H_{\mathcal{S}_A} = \infty$ .  $\square$

This last proposition finishes the proof of Theorem 6.1.

**6.2. Rotation sets for thin annular continua.** Let  $\mathcal{D} \subset \text{Diffeo}_+(\mathbb{T}^1)$  be the set of orientation-preserving circle diffeomorphisms with a totally disconnected non-wandering set. Note that this means  $f \in \mathcal{D}$  either has rational rotation number and a totally disconnected set of periodic points, or  $f$  is a Denjoy example (with irrational rotation number).

The aim of this section is to construct a family of examples of homeomorphisms  $f_{g,\alpha}$  of  $\mathbb{A}$ , parametrised by  $g \in \mathcal{D}$  and  $\alpha \in \mathbb{R}$  such that

- $f_{g,\alpha}$  leaves invariant some annular continuum  $A_{g,\alpha} \in \mathcal{A}_2$  containing at least one infinite spike, and
- $\rho_{A_{g,\alpha}}(F) = \text{conv}(\{\alpha, \rho(g)\})$ .<sup>6</sup>

This will prove Proposition 6.3.

For any  $t \in \mathbb{R}^+$ , let  $\mathcal{R}_t = \mathbb{R} \times \{t\}$  and define  $i : (0, +\infty) \rightarrow \mathbb{R}$  by  $i(x) = \frac{1}{x}$ . Further, let  $\mathcal{G} = \{L_p\}_{p \in \mathbb{R}}$  be the  $C^\infty$ -foliation of  $\mathbb{R} \times (0, +\infty]$  whose leaves are given by  $L_p = \text{gr}(i) + (p, 0)$  for every  $p \in \mathbb{R}$ , where  $\text{gr}(i) = \{(x, i(x)) \mid x > 0\}$ . Notice that for  $(x, y) \in \mathbb{R} \times (0, +\infty]$  the leave  $l_{(x,y)}$  through  $(x, y)$  is given by  $l_{(x,y)} = L_{p(x,y)}$  with  $p(x, y) = x - \frac{1}{y}$ .

Given  $g \in \mathcal{D}$ , we choose a lift  $G : \mathbb{R} \rightarrow \mathbb{R}$ . Then, we consider  $F_1 : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \times \mathbb{R}^+$  given by  $F_1(x, y) = (x + v(x, y), y)$  where  $v(x, y) = G(p(x, y)) - p(x, y)$ . Notice that  $F_1(l_{p(x,y)}) = l_{p(x,y)+v(x,y)}$  for every  $(x, y) \in \mathbb{R} \times \mathbb{R}^+$ . Hence,  $F_1$  preserves the set

$$\mathcal{T} := \bigcup_{p \in \pi^{-1}(\Omega(g))} L_p \cap (\mathbb{R} \times [0, 1]).$$

Further,  $F_1$  is a  $C^1$  diffeomorphism since  $p$  is  $C^\infty$  and  $G$  is  $C^1$ .

Let  $X : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \times \mathbb{R}^+$  be the vector field given by  $X(x, y) = (\alpha - v(F_1^{-1}(x, y)), t(x, y))$ , where  $t(x, y)$  is uniquely defined in order to have  $X(x, y) \in T_{(x,y)}l_{(x,y)}$ . Then we have that  $X$  is  $C^1$  (by the same argument that for  $F_1$ ), and  $\pi_1 \circ X(x, y)$  is constant over each leaf of

<sup>6</sup>Where  $\text{conv}(X)$  denotes the convex hull of  $X$ , and  $\rho(g)$  is the rotation number of  $g$ .

the foliation. Let  $H$  be the time one of the flow associated to  $X$ . We have that  $H$  preserves each leave of the foliation  $\mathcal{G}$ , and that  $\pi_1(H(x, y)) = \pi_1(x, y) + \alpha - v(F_1^{-1}(x, y))$  for all  $(x, y) \in \mathbb{R}^2$ .

We define  $F_2 : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \times \mathbb{R}^+$  by  $F_2(x, y) = H \circ F_1(x, y)$ . Thus we have that  $F_2$  is a  $C^1$  diffeomorphism which preserves each leave of  $\mathcal{G}$ , and that verifies  $\pi_1(F_2(x, y)) = (x, y) + \alpha$  (see Figure 6.5).

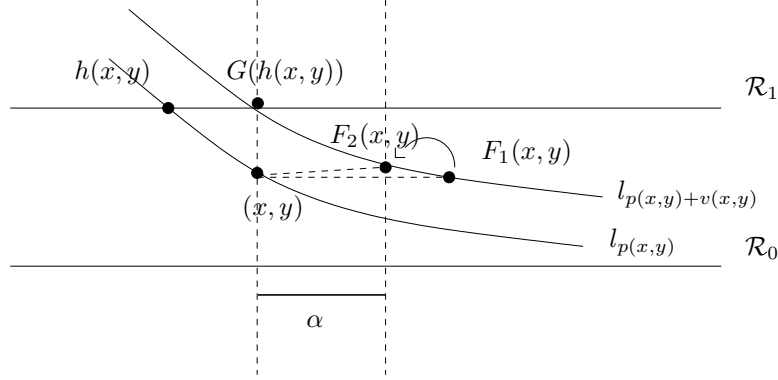


FIGURE 6.5. As the leaves of  $\mathcal{G}$  becomes horizontal as  $y$  tends to zero, we have that  $t(x, y)$  tends to zero as  $y$  goes to zero. Hence, so it does  $\pi_2(F_2(x, y) - (x, y))$ .

Due to the geometry of the foliation and the definition of the vector field, we have that horizontal lines which are close to  $\mathcal{R}$  get mapped in curves which remain close to  $\mathcal{R}$  (see Figure 6.5). Formally speaking we have that the family of maps  $\{\Delta_y : \mathbb{R}^+ \rightarrow \mathbb{R}^+\}_{y \in \mathbb{R}^+}$  defined by  $\Delta_y(x) = \pi_2(F_2(x, y))$ , tends uniformly to the constant zero as  $y$  goes to zero.

Thus we can consider  $y_0 \in (0, \frac{1}{3}]$  such that  $F_2(\mathcal{R}_{y_0}) \subset (0, \frac{1}{3}]$ . This implies that we can now define a  $T$ -invariant bijective function  $W : \mathbb{R} \times [y_0, \frac{2}{3}] \rightarrow \overline{\mathcal{U}^+(F_2(\{(x, y_0) \mid x \in \mathbb{R}\}))} \cap \mathcal{U}^-(\mathcal{R}_{\frac{2}{3}})$  given by  $W(x, y) = H_{s(y)} \circ F_1(x, y)$ , where  $H_s$  is the time  $s$  map of the flow associated to  $X$ , and  $s : [y_0, \frac{2}{3}] \rightarrow [0, 1]$  is a monotone decreasing continuous function from 1 to 0 such that  $H_{s(y)}$  defines an injective function in each set of the form  $l_p \cap [y_0, \frac{2}{3}]$ ,  $p \in \mathbb{R}$  (we omit the details for the construction of  $s$ ).

Then  $W$  leaves  $\mathcal{T}$  invariant and verifies  $W|_{\mathcal{R}_{y_0}} = F_2|_{\mathcal{R}_{y_0}}$  and  $W|_{\mathcal{R}_{\frac{2}{3}}} = F_1|_{\mathcal{R}_{\frac{2}{3}}}$ . Consequently, we can define a  $T$ -invariant homeomorphism  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  as follows.

$$F_{g,\alpha}(x, y) = \begin{cases} F_1(x, y) & \text{if } (x, y) \in \mathcal{U}^+(\mathcal{R}_{\frac{2}{3}}) \\ W(x, y) & \text{if } (x, y) \in \overline{\mathcal{U}^-(\mathcal{R}_{\frac{2}{3}})} \cap \mathcal{U}^+(\mathcal{R}_{y_0}) \\ F_2(x, y) & \text{if } (x, y) \in \mathcal{U}^+(\mathcal{R}) \cap \mathcal{U}^-(\mathcal{R}_{y_0}) \\ x + \alpha & \text{if } (x, y) \in \overline{\mathcal{U}^-(\mathcal{R})} \end{cases}$$

In order to see that  $F$  is a homeomorphism, we have to check that  $F$  is continuous for a point  $(x, 0) \in \mathbb{R}^2$ . However, due to construction, for any sequence  $(w_n)_{n \in \mathbb{N}}$  converging to  $(x, 0)$  we have that  $F_2(w_n) - w_n = (\alpha, d_n)$  with  $\lim_n d_n = 0$ . Thus  $\lim_n F_2(w_n) = (x + \alpha, 0)$ .

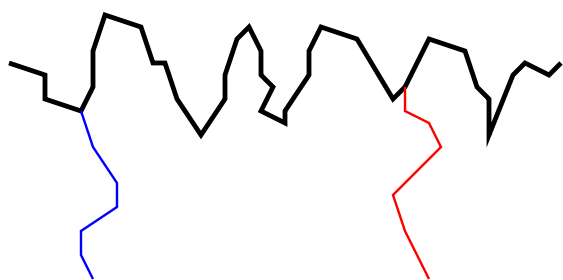
Hence,  $F_{g,\alpha}$  is a  $T$ -invariant homeomorphism of the upper half-plane, which can easily be extended to all of  $\mathbb{R}^2$  and thus defines a homeomorphism  $f_{g,\alpha} : \mathbb{A} \rightarrow \mathbb{A}$ . Furthermore  $f_{g,\alpha}$  leaves invariant the essential thin annular continuum given by  $A_{g,\alpha} := \pi(\mathcal{R}_0 \cup \mathcal{T}) = \Omega(g) \times \{1\}$ , which has at least one infinite spike and whose circlod  $C_{A_{g,\alpha}} = \pi(\mathcal{R}_0) = \mathbb{T}^1 \times \{0\}$  is compactly generated.

Finally, it follows from the definition of  $f$  that points in  $A \cap \pi(\mathcal{R}_1)$  have a unique rotation vector  $(\rho(G), 0)$ , and points in  $C_A$  have rotation vector  $(\alpha, 0)$ . Hence  $\rho_A(F) \supset I = \text{conv}(\alpha, \rho(g))$ . Furthermore, given any point  $z \in A \setminus (C_A \cup \pi(\mathcal{R}_1))$ , we have by construction that  $\pi_1(F^n(z) - z) \in \text{conv}(n \cdot \alpha, [G^n(h(z)) - h(z)])$ . This implies that  $\rho_A(F) = I$ .

## REFERENCES

- [1] R. Johnson and J. Moser. The rotation number for almost periodic potentials. *Commun. Math. Phys.*, 4:403–438, 1982.
- [2] D.H. Perkel, J.H. Schulman, T.H. Bullock, G.P. Moore, and J.P. Segundo. Pacemaker neurons: Effects of regularly spaced synaptic input. *Science*, 145:61–63, 1964.
- [3] M. Misiurewicz and K. Ziemian. Rotation sets for maps of tori. *J. Lond. Math. Soc.*, 40:490–506, 1989.
- [4] J. Llibre and R.S. MacKay. Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity. *Ergodic Theory Dyn. Syst.*, 11:115–128, 1991.
- [5] J. Franks. Realizing rotation vectors for torus homeomorphisms. *Trans. Am. Math. Soc.*, 311(1):107–115, 1989.
- [6] T. Jäger. Linearisation of conservative toral homeomorphisms. *Invent. Math.*, 176(3):601–616, 2009.
- [7] T. Jäger. The concept of bounded mean motion for toral homeomorphisms. *Dyn. Syst.*, 24(3):277–297, 2009.
- [8] A. Koropecski and F. Armando Tal. Area-preserving irrotational diffeomorphisms of the torus with sublinear diffusion. Preprint 2012, [arXiv:1206.2409](#).
- [9] A. Koropecski and F. Armando Tal. Bounded and unbounded behavior for area-preserving rational pseudo-rotations. Preprint 2012, [arXiv:1207.5573](#).
- [10] J. Franks and M. Misiurewicz. Rotation sets of toral flows. *Proc. Am. Math. Soc.*, 109(1):243–249, 1990.
- [11] P. Davalos. On torus homeomorphisms whose rotation set is an interval. Preprint 2012, [arXiv:1111.2378](#).
- [12] N. Guelman, A. Koropecski, and F. Armando Tal. A characterization of annularity for area-preserving toral homeomorphisms. Preprint 2012, [arXiv:1211.5044](#).
- [13] M. Misiurewicz and K. Ziemian. Rotation sets and ergodic measures for torus homeomorphisms. *Fundam. Math.*, 137(1):45–52, 1991.
- [14] P. Le Calvez. Propriétés dynamiques des difféomorphismes de l’anneau et du tore. *Astérisque*, 204, 1991.
- [15] J. Kwapisz. Combinatorics of torus diffeomorphisms. *Ergodic Theory Dyn. Syst.*, 23:559–586, 2003.
- [16] P. Le Calvez. Une version feuilleté équivariante du théorème de translation de Brouwer. *Publ. Math. Inst. Hautes Études Sci.*, 102:1–98, 2005.
- [17] T. Jäger. Elliptic stars in a chaotic night. *J. Lond. Math. Soc.*, 84(3):595–611, 2011.
- [18] A. Koropecski and F. Armando Tal. Strictly toral dynamics. Preprint 2012, [arXiv:1201.1168](#).
- [19] J. Kwapisz. Poincaré rotation number for maps of the real line with almost periodic displacement. *Nonlinearity*, 13(5):1841, 2000.
- [20] M. Herman. Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d’un théorème d’Arnold et de Moser sur le tore de dimension 2. *Comment. Math. Helv.*, 58:453–502, 1983.
- [21] A. Koropecski. *On the dynamics of torus homeomorphisms*. PhD thesis, IMPA (Brasil), 2007.
- [22] P. Le Calvez. Personal communication. Not published.
- [23] C. Carathéodory. Über die Begrenzung einfach zusammenhängender Gebiete. *Math. Ann.*, 73(3):323–370, 1913.
- [24] C. Carathéodory. Über die gegenseitige Beziehung der Ränder bei der konformen Abbildung des Inneren einer Jordanschen Kurve auf einen Kreis. *Math. Ann.*, 73(2):305–320, 1913.
- [25] P. Le Calvez A. Koropecski and M. Nassiri. Prime ends rotation numbers and periodic points. Preprint 2012, [arXiv:1206.4707](#).
- [26] P. Le Calvez. Propriétés des attracteurs de Birkhoff. *Ergodic Theory Dyn. Syst.*, 8(2):241–310, 1988.
- [27] R.H. Bing. Concerning hereditarily indecomposable continua. *Pac. J. Math.*, 1:43–51, 1951.
- [28] M. Handel. A pathological area preserving  $C^\infty$  diffeomorphism of the plane. *Proc. Am. Math. Soc.*, 86(1):163–168, 1982.
- [29] M. Herman. Construction of some curious diffeomorphisms of the Riemann sphere. *J. Lond. Math. Soc.*, 34:375–384, 1986.
- [30] R. Walker. Periodicity and Decomposability of Basin Boundaries with Irrational Maps on Prime Ends. *Transactions of the A.M.S.*, 324(1):303–317, 1991.
- [31] A. Hatcher. *Algebraic topology*. Cambridge University Press, 2002.
- [32] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1997.
- [33] J. Stark and R. Sturman. Semi-uniform ergodic theorems and applications to forced systems. *Nonlinearity*, 13(1):113–143, 2000.
- [34] R. Potrie. Recurrence of non-resonant torus homeomorphisms. *Proceedings of the A.M.S.*, 140:3973–3981, 2012.
- [35] M. Cartwright and J. Littlewood. Some fixed point theorems. *Ann. of Math.*, 54(2):1–37, 1951.
- [36] R.H. Bing. A homogeneous indecomposable plane continuum. *Duke Math. J.*, 15:729–742, 1948.
- [37] V. Guillemin and A. Pollack. *Differential Topology*. Prentice-Hall, 1974.

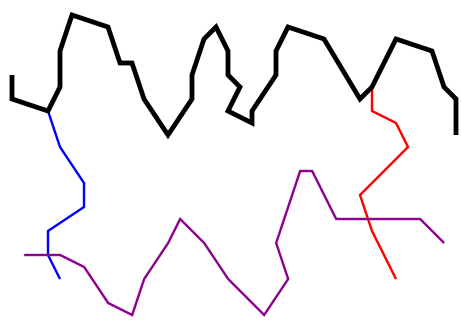
CA



C1

C2

L



C1

K

C2